Computer Graphics

Spectral Analysis

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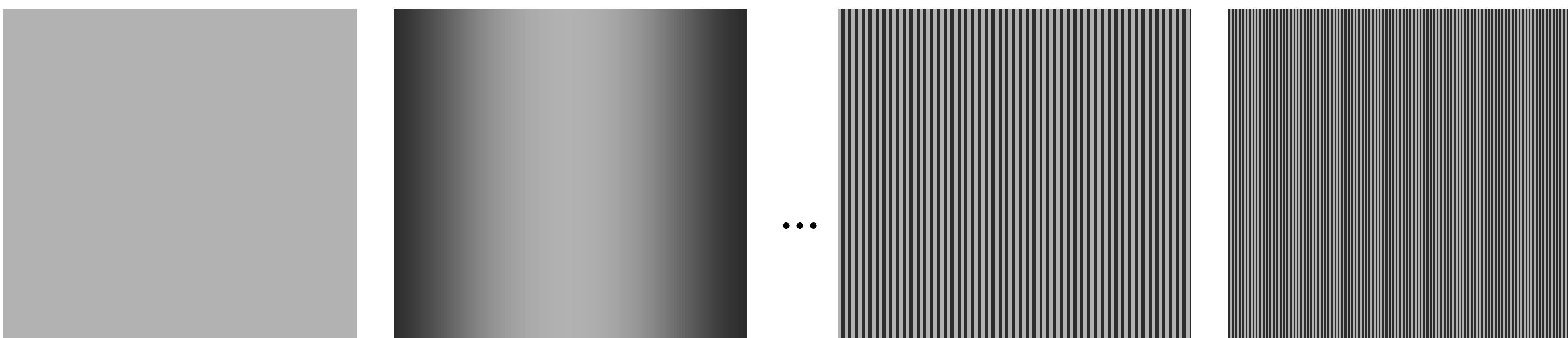
Spatial Frequency (of an image)

- **Frequency**
  - Inverse of period length of some structure in an image
  - Unit [1/pixel]
- **Lowest frequency**
  - Image average
- **Highest representable frequency**
  - Nyquist frequency (1/2 the sampling frequency)
  - Defined by half the image resolution
- **Phase allows shifting of the pattern**



Fourier Transformation

- **Any absolute integrable function *f(x)* can be expressed as an integral over sine and cosine waves:**

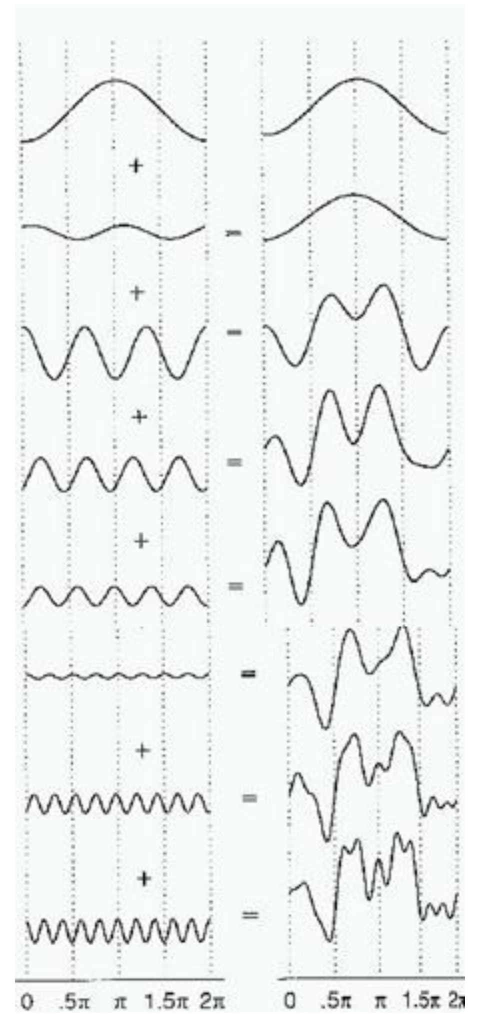
Analysis: $$F(k) = F_x[f(x)](k) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi kx} dx$$

Synthesis: $$f(x) = F_x^{-1}[F(k)](x) = \int_{-\infty}^{\infty} F(k) e^{i2\pi kx} dk$$

- **Representation via complex exponential**
  - $e^{ix} = \cos(x) + i \sin(x)$ (see Taylor expansion)
- **Division into even and odd parts**
  - Even: f(x) = f(-x) (symmetry about *y* axis)
  - Odd: f(x) = -f(-x) (symmetry about origin)

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] = E(x) + O(x)$$

- **Transform of each part**
  - Even: *cosine* only; odd: *sine* only



Analysis & Synthesis

Symetric integral ([-a, a]) over an odd function is zero

- **Analysis**

$$F(k) = \int_{-\infty}^{\infty} f(x)\,(\cos(-2\pi kx) + i\sin(-2\pi kx))dx = b(k) + i\,a(k)$$

  - Even term

$$b(k) = \int_{-\infty}^{\infty} f(x)\cos(2\pi kx)\,dx = \int_{-\infty}^{\infty} (E(x) + O(x))\cos(2\pi kx)\,dx = \int_{-\infty}^{\infty} E(x)\cos(2\pi kx)\,dx$$

  - Odd term

$$a(k) = \int_{-\infty}^{\infty} f(x)\sin(2\pi kx)\,dx = \int_{-\infty}^{\infty} (E(x) + O(x))\sin(2\pi kx)\,dx = \int_{-\infty}^{\infty} O(x)\sin(2\pi kx)\,dx$$

- **Synthesis**

$$f(x) = \int_{-\infty}^{\infty} F(k)(\cos(2\pi kx) + i\sin(2\pi kx))\,dk = E(x) + O(x)$$

  - Even term

$$E(x) = \int_{-\infty}^{\infty} F(k)\cos(2\pi kx)\,dk = \int_{-\infty}^{\infty} (b(k) - i\,a(k))\cos(2\pi kx)\,dk = \int_{-\infty}^{\infty} b(k)\cos(2\pi kx)\,dk$$

  - Odd term

$$O(x) = \int_{-\infty}^{\infty} F(k)\,i\sin(2\pi kx)\,dk = \int_{-\infty}^{\infty} (b(k) - i\,a(k))\,i\sin(2\pi kx)\,dk = \int_{-\infty}^{\infty} a(k)\sin(2\pi kx)\,dk$$

Spatial vs. Frequency Domain

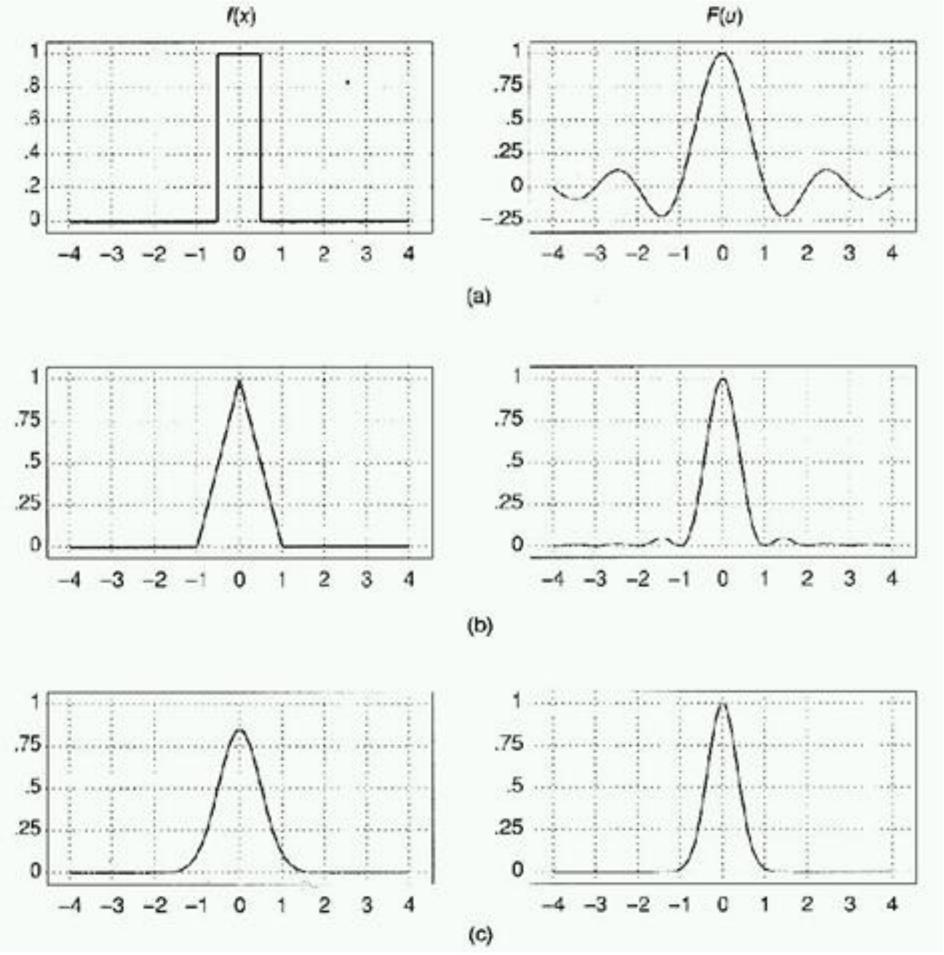
- **Important basis functions:**
  - Box ↔ (normalized) *sinc*

$$\text{sinc}(x) = \frac{\sin(x\pi)}{x\pi}$$

$$\text{sinc}(0) = 1$$

$$\int \text{sinc}(x)dx = 1$$

    - Negative values
    - Infinite support
  - Tent ↔ *sinc*^{2}
    - Tent == Convolution of box function with itself
  - Gaussian ↔ Gaussian
    - Inverse width

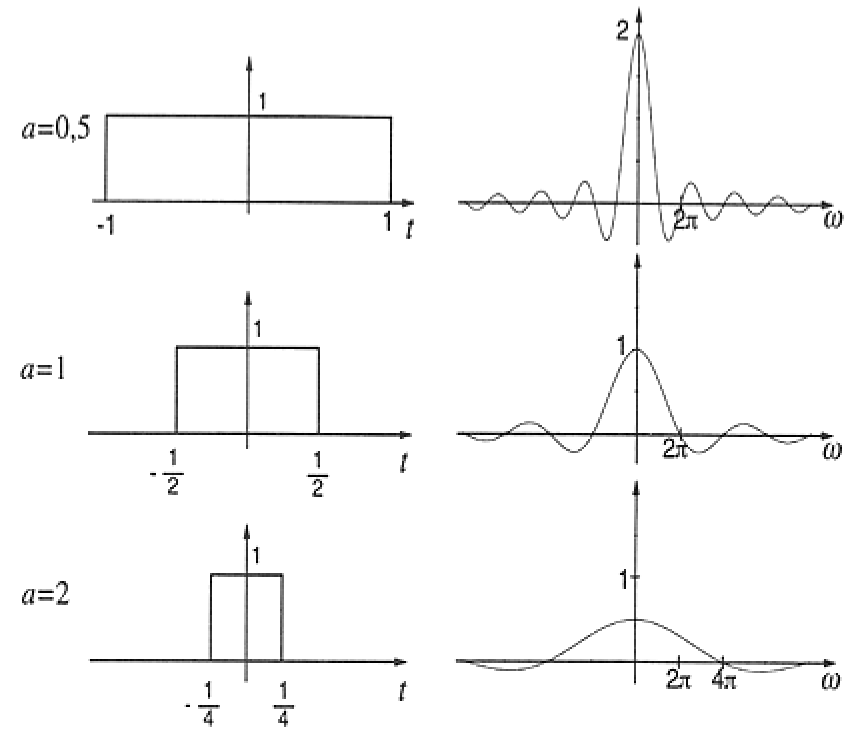


Spatial vs. Frequency Domain

- **Transform behavior**
- **Example: Fourier transform of a box function**

$$\mathrm{rect}(at) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{|a|}\,\mathrm{si}\left(\frac{\omega}{2a}\right).$$

  - Wide box → narrow *sinc*
  - Box → *sinc*
  - Narrow box → wide *sinc*



Fourier Transformation

- **Periodic in space ⇔ discrete in frequency (vice ver.)**
  - Any periodic, continuous function can be expressed as the sum of an (infinite) number of sine or cosine waves:

$$f(x) = \Sigma_k \, a_k \sin(2\pi*k*x) + b_k \cos(2\pi*k*x)$$

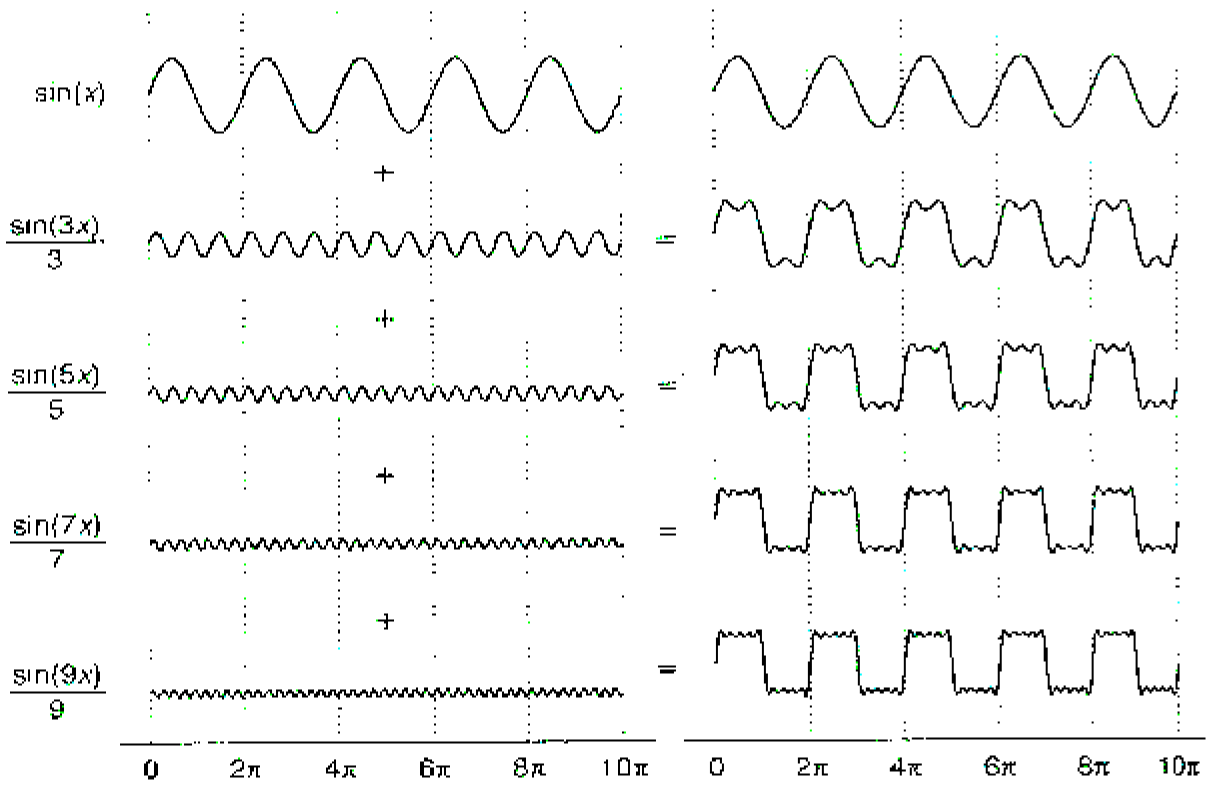
  - Any finite interval can be made periodic by concatenation with itself
- **Decomposition of signal into different frequency bands: Spectral Analysis**
  - Frequency band: k
    - $k = 0$: mean value
    - $k = 1$: function period, lowest possible frequency
    - $k = 1.5$? : not possible, periodic function, e.g. $f(x) = f(x+1)$
    - k_{max} ? : band limit, no higher frequency present in signal
  - Fourier coefficients: a_k, b_k (real-valued, as before)
    - Even function $f(x) = f(-x)$: $a_k = 0$
    - Odd function $f(x) = -f(-x)$: $b_k = 0$

Fourier Synthesis Example

- **Square wave: periodic, uneven function**

$$f(x) = 0.5 \quad \forall\, 0 < (x \bmod 2\pi) < \pi$$
$$= -0.5 \quad \forall\, \pi < (x \bmod 2\pi) < 2\pi$$
$$a_k = \int \sin(2\pi kx) f(x)\, dx \qquad f(x) = \sum_k a_k \sin(2\pi kx)$$

- $a_0 = 0$
- $a_1 = 1$
- $a_2 = 0$
- $a_3 = 1/3$
- $a_4 = 0$
- $a_5 = 1/5$
- $a_6 = 0$
- $a_7 = 1/7$
- $a_8 = 0$
- $a_9 = 1/9$
- …



Discrete Fourier Transform

- **Equally-spaced function samples (*N* samples)**
  - Function values known only at discrete points, e.g.
    - Idealized physical measurements
    - Pixel positions in an image!
      - Represented via sum of Delta distribution (Fourier integrals → sums)
- **Fourier analysis**

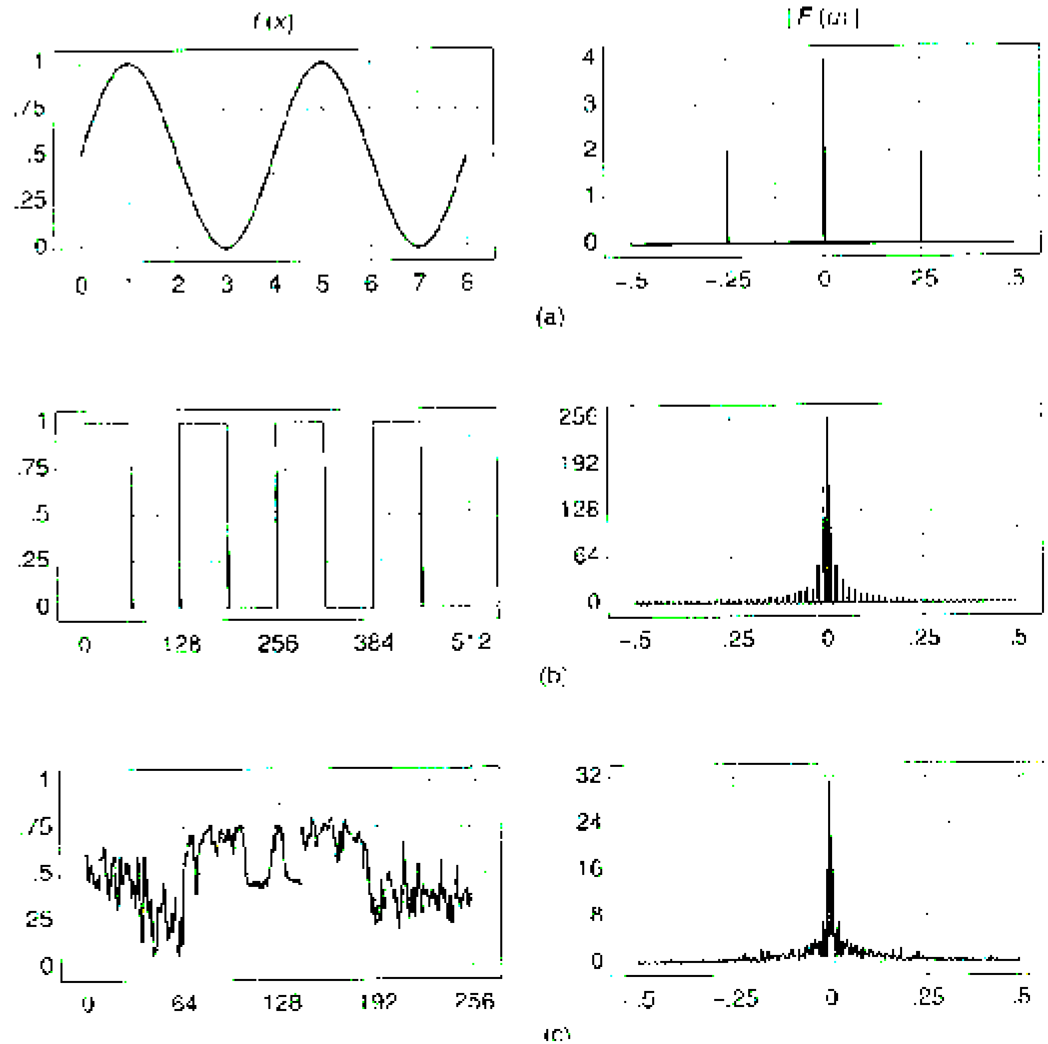
$$a_k = \sum_i \sin\left(\frac{2\pi k i}{N}\right) f_i$$

$$b_k = \sum_i \cos\left(\frac{2\pi k i}{N}\right) f_i$$

  - Sum over all *N* measurement points
  - $k = 0, 1, 2, \ldots$? Highest possible frequency?
    - Nyquist frequency: highest frequency that can be represented
    - Defined as 1/2 the sampling frequency
    - Sampling rate *N:* determined by image resolution (pixel size)
    - 2 samples / period ↔ 0.5 cycles per pixel ⇒ $k_{max} \leq N / 2$

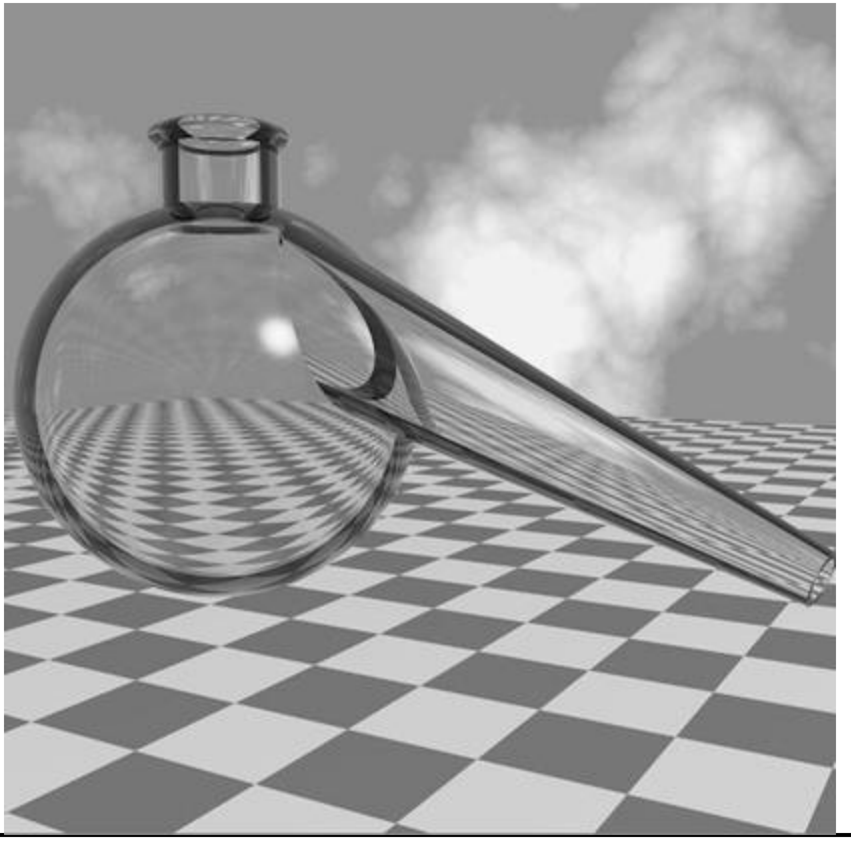
Spatial vs. Frequency Domain

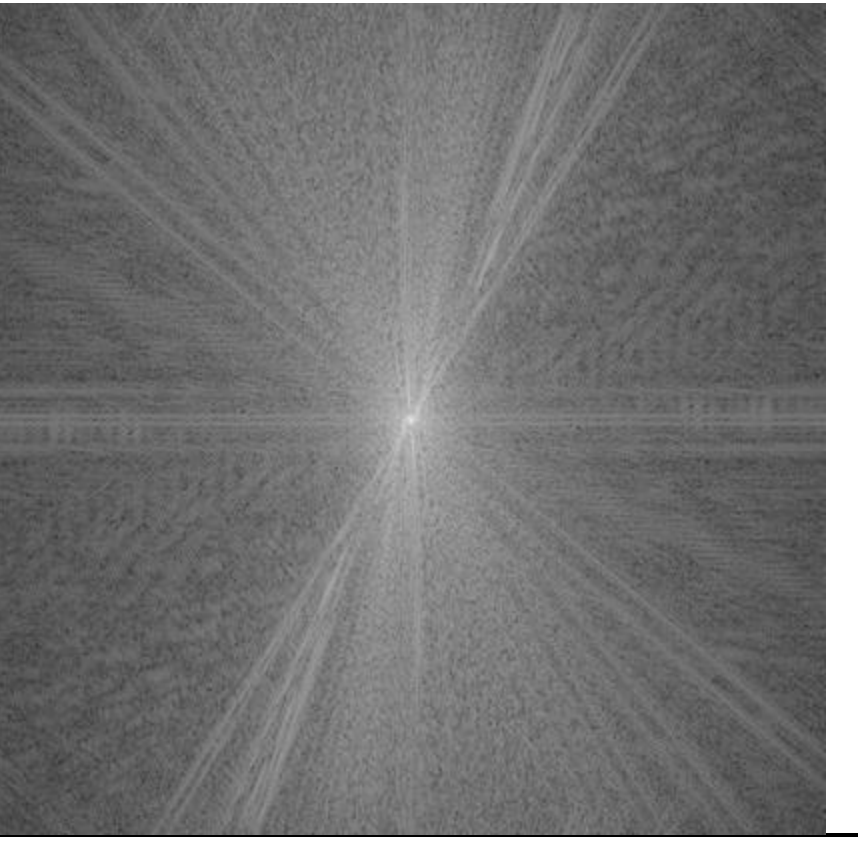
- **Examples (pixels vs. cycles per pixel)**
  - Sine wave with positive offset
  - Square wave with offset
  - Scanline of an image



2D Fourier Transform

- **2 separate 1D Fourier transformations along *x* and *y* directions**
- **Discontinuous edge → line in orthogonal direction in Fourier domain !**

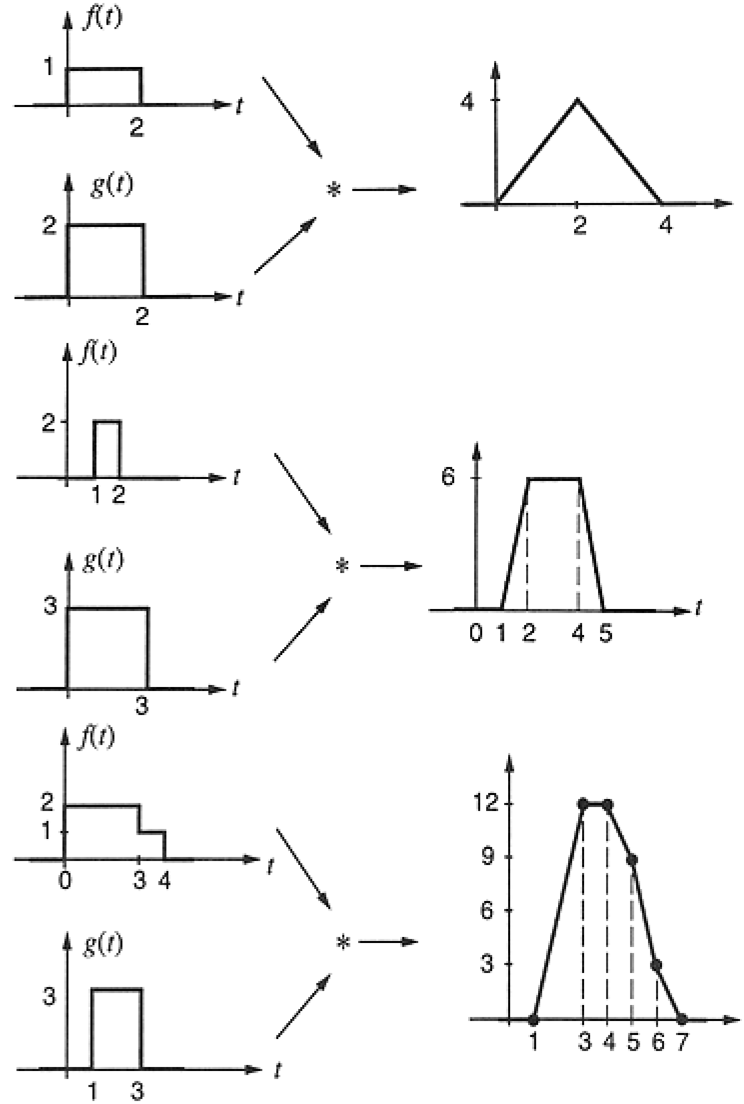




Convolution

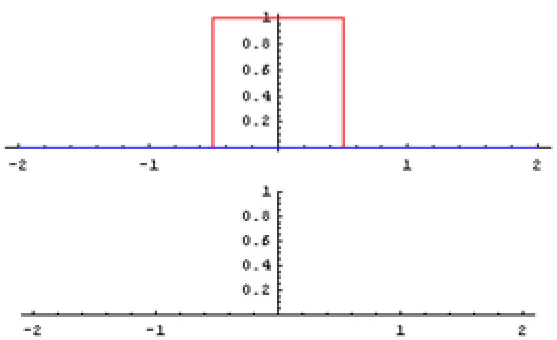
$$(f \otimes g)(x) = \int_{-\infty}^{\infty} f(\tau)g(x-\tau)d\tau$$

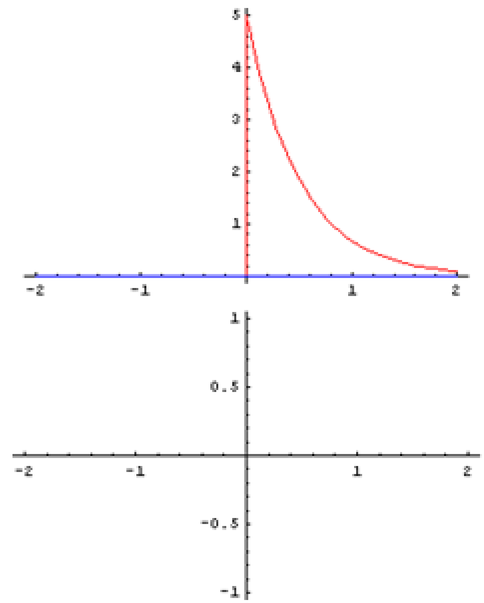
- **Two functions *f*, *g***
- **Shift one (reversed) function against the other by *x***
- **Multiply function values**
- **Integrate across overlapping region**
- **Numerical convolution: expensive operation**
  - For each *x*:
    integrate over non-zero domain



Convolution

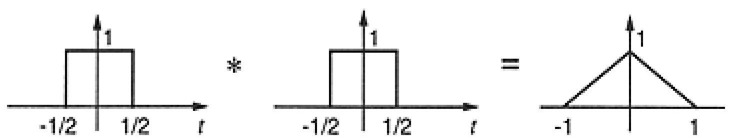
- **Examples**





Convolution Theorem

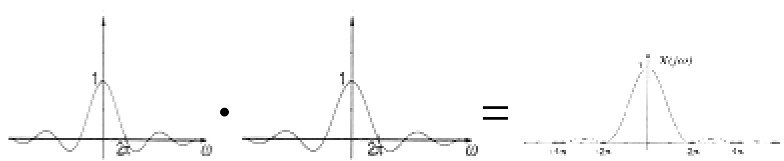
- **Convolution in image domain**
  → Multiplication in Fourier domain
- **Convolution in Fourier domain**
  → Multiplication in image domain
- **Multiplication in transformed Fourier domain may be cheaper than direct convolution in image domain !**



$$x(t) = \begin{cases} t+1 & \text{für } -1 \le t < 0 \\ -t+1 & \text{für } 0 \le t < 1 \\ 0 & \text{sonst} \end{cases}$$

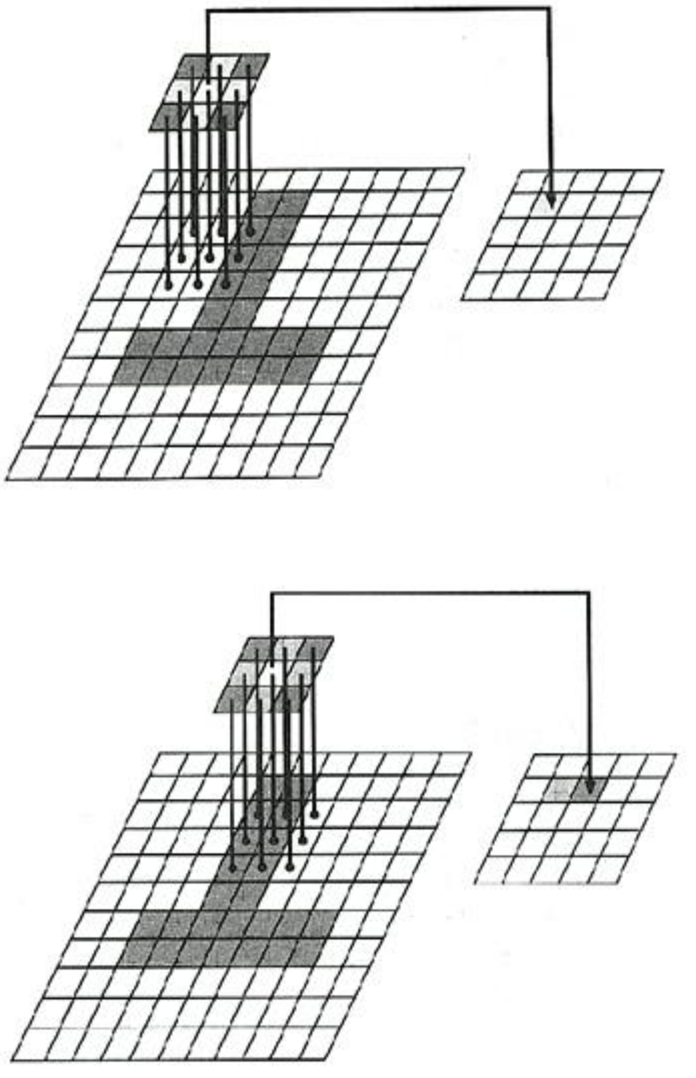
$$\mathrm{rect}(t) \ast \mathrm{rect}(t) = x(t)$$

$$\mathrm{si}\left(\frac{\omega}{2}\right) \cdot \mathrm{si}\left(\frac{\omega}{2}\right) = X(j\omega) = \mathrm{si}^2\left(\frac{\omega}{2}\right).$$



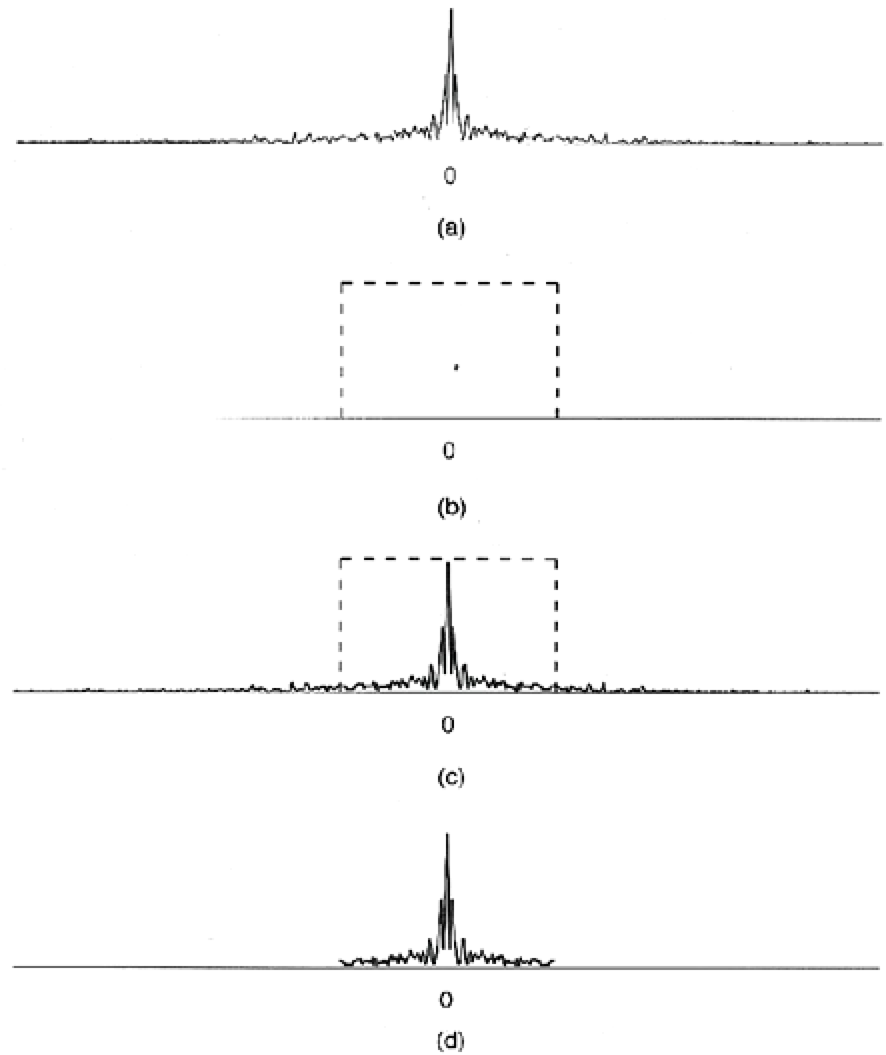
Convolution and Filtering

- **Technical realization**
  - In image domain
  - Pixel mask with weights
- **Problems (e.g. *sinc*)**
  - Large filter support
    - Large mask
    - A lot of computation
  - Negative weights
    - Negative light?



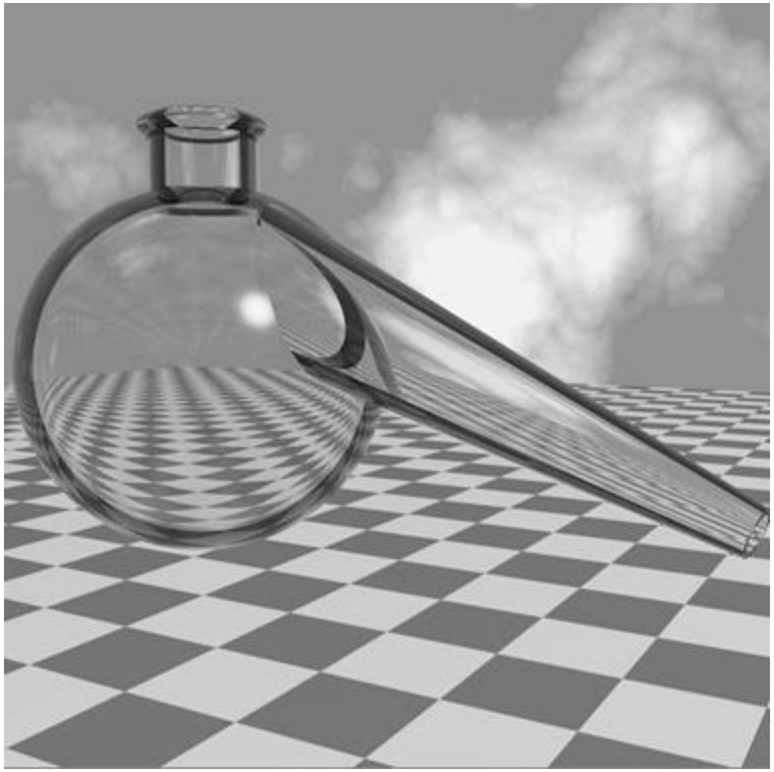
Filtering

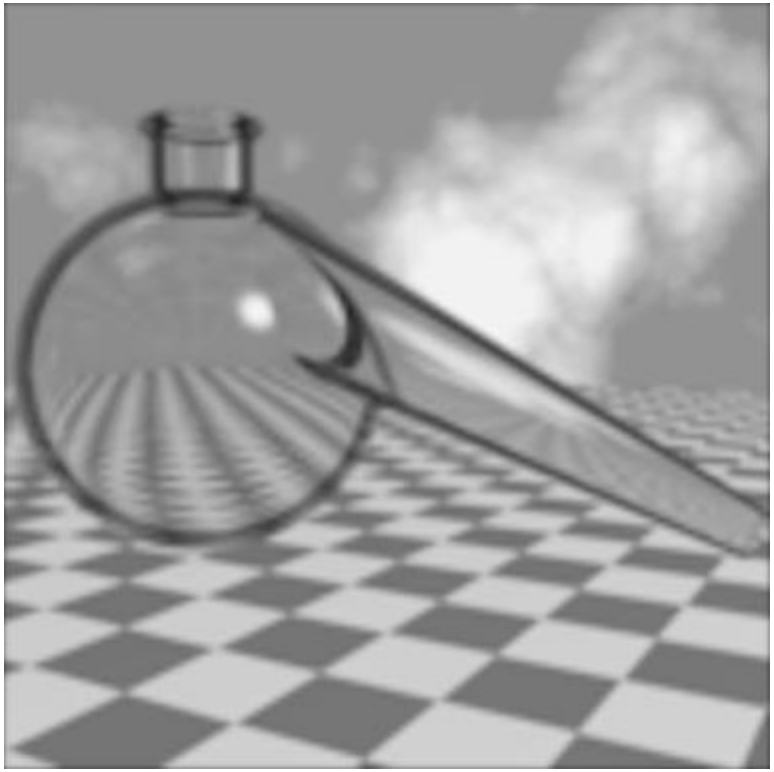
- **Ideal low-pass filter**
  - Multiplication with box in frequency domain
  - Convolution with *sinc* in spatial domain
- **Ideal high-pass filter**
  - Multiplication with (1 - box) in frequency domain
  - Only high frequencies
- **Ideal band-pass filter**
  - Combination of wide low-pass and narrow high-pass filter
  - Only intermediate frequencies



Low-Pass Filtering

- **“Blurring”**





High-Pass Filtering

- **Enhances discontinuities in image**
  - Useful for edge detection