Computer Graphics

Introduction to Monte Carlo Rendering

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Overview

- Introduction to Discrete Probability Theory
- Introduction to Continuous Probability Theory
- Monte Carlo Integration : A practical approach

Introduction to Discrete Probability Theory

Motivation: Game of dice
We throw two fair dice, one red and one green.

(a) What is the set of possible results?
(b) Which results give a total of 6?
(c) Which results give a total of 12?
(d) Which results give an odd total?
(e) Which are the probabilities of the events (b),(c),(d)?

Calculation of probabilities

We can try to calculate the probabilities of events such as (b), (c) and (d) by throwing the dice numerous times and letting

$$\text{probability of an event} = \frac{\text{\# of times event takes place}}{\text{\# experiments carried out}}$$

This is an empirical rather than a mathematical answer!

Probability space

A **probability space** $(\Omega, \mathcal{F}, P)$ is a mathematical object associated with a random experiment comprising:

1. a set Ω, the **sample space** (universe), which contains all the possible outcomes (or results) ω of the experiment;
2. a collection $\mathcal{F}$ of subsets of Ω. These subsets are called **events**, and $\mathcal{F}$ is called the **event space**;
3. a function $P : \mathcal{F} \to [0, 1]$ called a **probability distribution**, which associates a probability $\mathrm{P}\left(A\right) \in [0, 1]$ to each $A \in \mathcal{F}$.

For simple examples with finite Ω, we often choose Ω so that each $\omega \in \Omega$ is equiprobable: If $\mathrm{P}\left(\omega\right) = \frac{1}{|\Omega|}$, for every $\omega \in \Omega$, then $\mathrm{P}\left(A\right) = \frac{|A|}{|\Omega|}$, for every $A \subset \Omega$.

Example 1. Sample Space

What is the sample space of the following experiments:

(a) I toss a coin.

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Solution of Example 1

(a) $\Omega = \{\omega_1, \omega_2\}$, where ω_1 and ω_2 represents Tail and Head respectively.

(b) $\Omega = \{\omega_1, ..., \omega_{36}\}$, representing all 36 different possibilities.

Example 2. Event Space

$\mathcal{F} = \{A, B\}$ is a set of subsets of Ω which represents the events of interest. For the experiment “I roll two fair dice, one red and one green”, what are the events:

(a) A: the red die shows a 4,

(b) B: the total is odd

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Solution of Example 2

If we define $\Omega = \{(r, g) : r, g = 1, ..., 6\}$, where r and g represent the red and green die respectively, we can write:

(a) $A = \{(4, g), g = 1, ...6\}$

(b) $B = \{(1, 2), (1, 4), (1, 6), ..., (6, 1), (6, 3), (6, 5)\}$

Set operations

Given two sets A and B we can define the following operations:

$A \cap B$	intersection between set A and set B
$A \cup B$	union between set A and set B
$A \setminus B$	set A without the elements of set B
$A \subset B$	set A is a subset of set B
A^c	complement of set A

Properties of Probability Distributions

Given A and B two events of the probability space $(\Omega, \mathcal{F}, \mathrm{P})$, the following properties are true:

- $\mathrm{P}(\emptyset) = 0$
- $\mathrm{P}(\Omega) = 1$
- $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$.
  If $A \setminus B = \emptyset$, then $\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B)$
- if $A \subset B$, then $\mathrm{P}(A) \leq \mathrm{P}(B)$, and $\mathrm{P}(B \setminus A) = \mathrm{P}(B) - \mathrm{P}(A)$

Example 3. Probability Distributions

We roll two fair dice, one red and one green. What is the probability of

(a) the result of the red die is 4, and the total sum is 9?

(b) the result of the red die is 4, or the total sum of the dice is 9?

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Solution of Example 3

$\mathrm{P}(A) = \mathrm{P}(\text{“red die is 4”}) = \frac{6}{36}$, and $\mathrm{P}(B) = \mathrm{P}(\text{“sum is 9”}) = \frac{4}{36}$.
Hence

(a) $\mathrm{P}(A \cap B) = \frac{1}{36}$

(b) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = \frac{6}{36} + \frac{4}{36} - \frac{1}{36} = \frac{9}{36} = \frac{1}{4}$

Discrete Probability Theory

green die

red die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Conditional Probability Distributions

Let A and B be events of the probability space $(\Omega, \mathcal{F}, P)$, such that $\mathrm{P}(B) > 0$. Then **the conditional probability of A given B** is

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$$

If $\mathrm{P}(B) = 0$, we adopt the convention
$\mathrm{P}(A \cap B) = \mathrm{P}(A \mid B)\,\mathrm{P}(B)$, so both sides are equal to zero.

Independence

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Two events $A, B \in \mathcal{F}$ are **independent** (we write $A \perp\!\!\!\perp B$) iff

$$\mathrm{P}\left(A \cap B\right) = \mathrm{P}\left(A\right) \mathrm{P}\left(B\right)$$

In compliance with our intuition, this implies that

$$\mathrm{P}\left(A \mid B\right) = \frac{\mathrm{P}\left(A \cap B\right)}{\mathrm{P}\left(B\right)} = \frac{\mathrm{P}\left(A\right) \mathrm{P}\left(B\right)}{\mathrm{P}\left(B\right)} = \mathrm{P}\left(A\right),$$

and by symmetry $\mathrm{P}\left(B \mid A\right) = \mathrm{P}\left(B\right)$.

Example 4. Independence

A pack of 52 cards is well-shuffled, and one card is randomly picked.

(a) Are the events A "the card is an ace" and H "the card is a heart" independent?

(b) What can we say about the events A and K "the card is a king"?

Solution of Example 4

The sample space Ω consists of the 52 cards, which are equiprobable ($\mathrm{P}\left(\text{“any card”}\right) = \frac{1}{52}$).

(a) $\mathrm{P}\left(A\right) = \frac{4}{52} = \frac{1}{13}$ and $\mathrm{P}\left(H\right) = \frac{13}{52} = \frac{1}{4}$, and $\mathrm{P}\left(A \cap H\right) = \frac{1}{52} = \mathrm{P}\left(A\right)\mathrm{P}\left(H\right)$, so A and H are independent.

(b) A card cannot be simultaneously an ace and a king, meaning $\mathrm{P}\left(A \cap K\right) = 0 \neq \mathrm{P}\left(A\right)\mathrm{P}\left(K\right)$, so these two events **are not independent**.

Random Variables

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A **random variable (rv)** $X : \Omega \mapsto \mathbb{R}$ is a function from the space sample Ω taking values in the real numbers $\mathbb{R}$. The set of values taken by X,

$$D_X = \{x \in \mathbb{R} : \exists \omega \in \Omega \text{ such that } X(\omega) = x\}$$

is called the **support** of X. If D_X is **countable**, then X is a **discrete random variable**.

Example 6. Random Variables I

We roll two fair dice, one red and one green. Let X be the total of the sides facing up. Find all possible values of X and the corresponding probabilities.

Solution of Example 6

Draw a grid. X takes values in $D_X = \{2, 3..., 11, 12\}$, and so is clearly a discrete random variable. By symmetry, the 36 points in Ω are equally likely, so, for example,

$$\mathrm{P}\left(X = 3\right) = P\left(\{(1,2),(2,1)\}\right) = \frac{2}{36}$$

Thus, the probabilities for $\{2, 3, 4..., 11, 12\}$ are respectively

$$\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$$

Example 7. Random Variables II

We toss a coin repeatedly and independently. Let X be the random variable representing the number of throws until we first get heads. Calculate:

(a) $\mathrm{P}\left(X = 3\right)$

(b) $\mathrm{P}\left(X = 15\right)$

(c) $\mathrm{P}\left(X \leq 3.5\right)$

(d) $\mathrm{P}\left(1.7 \leq X \leq 3.5\right)$

Solution of Example 7 (Part I)

X takes value in $\{1, 2, 3, ...\} = \mathbb{N}$, and so is a **discrete random variable** with **countable support**.
Let p be the probability of **success** (head) and $(1 - p)$ the probability of **failure** (tail) during a toss:

(a) The event $X = 3$ corresponds to two failures followed by a success, giving $\mathrm{P}(X = 3) = (1 - p)^2 p$ by **independence of the successive trials**.

(b) Likewise, $\mathrm{P}(X = 15) = (1 - p)^{14} p$, with 14 failures followed by a success.

Solution of Example 7 (Part II)

(c) We can compute the probability as follows:

$$\begin{aligned}
\mathrm{P}(X \leq 3.5) &= \mathrm{P}(X \leq 3) + \mathrm{P}(3 < X \leq 3.5) \\
&= p + (1-p)p + (1-p)^2 p \\
&= 1 - \mathrm{P}(X > 3) \\
&= 1 - (1-p)^3
\end{aligned}$$

(d) In this case, only two or three tosses are possible:

$$\begin{aligned}
\mathrm{P}(1.7 \leq X \leq 3.5) &= \mathrm{P}(X = 2) + \mathrm{P}(X = 3) \\
&= (1-p)p + (1-p)^2 p \\
&= p(1-p)(1+1-p) \\
&= p(1-p)(2-p)
\end{aligned}$$

Probability mass functions

A random variable X associates probabilities to subsets of $\mathbb{R}$. In particular, when X is discrete, we have:

$$A_x = \{\omega \in \Omega : X(\omega) = x\},$$

and we can define the **probability mass function (PMF)** of a discrete random variable X as:

$$f_X(x) = \mathrm{P}\left(X = x\right) = \mathrm{P}\left(A_x\right), x \in \mathbb{R}.$$

It has two properties:

(a) $f_X(x) \geq 0$, and it is only positive for $x \in D_X$, where D_X is the image of the function X, i.e., the **support** of f_X;

(b) the total probability $\sum_{\{i : x_i \in D_X\}} f_X(x_i) = 1$

Example 8. Probability mass functions

We roll two fair dice, one red and one green. Let X be the total of the sides facing up. Compute the probability mass function of the variable X and represent it graphically.

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Solution of Example 8

The x axis should represents all the values X can take, the support of f_X, while the $y = f_X(x)$ axis represent the corresponding discrete probabilities:
$\mathrm{P}\left(X = 2\right), \mathrm{P}\left(X = 3\right), ..., \mathrm{P}\left(X = 11\right), \mathrm{P}\left(X = 12\right)$.

Probability Distributions: Geometric Distribution

A **geometric** random variable X has PMF

$$f_X(x) = p(1-p)^{x-1}, \quad \text{with} \;\; x = 1, 2, ..., N \text{ and } \;\; 0 \leq p \leq 1$$

We write $X \sim \mathrm{Geom}(p)$, and we call **p** the **success probability**.

Note: This distribution models the waiting time X until a first successful event in a series of **independent** trials having the **same success probability**.

Probability Distributions: Discrete Uniform Distribution

A **discrete uniform** random variable X has PMF

$$f_X(x) = \frac{1}{b-a+1}, \quad \text{with} \quad x = a, a+1, ..., b, \quad a < b, \quad a, b \in \mathbb{Z}$$

We write $U \sim DU(a, b)$.

Note: This definition generalises the outcome of die-throw, which corresponds to the $DU(1, 6)$ distribution.

Cumulative distribution function

The **cumulative distribution function (CDF)** of a random variable X is:

$$F_X(x) = \mathrm{P}\left(X \leq x\right), \quad x \in \mathbb{R}.$$

If X is discrete, we can write

$$F_X(x) = \sum_{\{x_i \in D_X : x_i \leq x\}} \mathrm{P}\left(X = x_i\right),$$

which is a step function with jumps at the points of support D_X of $f_X(x)$.
When there is no risk of confusion, we write $F \equiv F_X$

Example 9. Cumulative distribution function

Give the support and the probability mass and cumulative distribution functions of a geometric random variable.

Solution of Example 9

The support is $D = \mathbb{N}$, and for $x \geq 1$ we have

$$\mathrm{P}\left(X \leq x\right) = \sum_{r=1}^{\lfloor x \rfloor} p(1-p)^{r-1},$$

so we need to sum a geometric series with common ratio $1-p$, giving

$$\mathrm{P}\left(X \leq x\right) = \frac{p\{1-(1-p)^{\lfloor x \rfloor}\}}{1-(1-p)} = 1-(1-p)^{\lfloor x \rfloor}$$

Thus

$$F_X(x) = \mathrm{P}\left(X \leq x\right) = \begin{cases} 0, & x < 1 \\ 1-(1-p)^{\lfloor x \rfloor}, & x \geq 1 \end{cases}$$

Properties of a cumulative distribution function

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \mapsto \mathbb{R}$ a random variable. Its cumulative distribution function F_X satisfies:

(a) $\lim_{x \to -\infty} F_X(x) = 0$;

(b) $\lim_{x \to +\infty} F_X(x) = 1$;

(c) F_X is non-decreasing, so $F_X(x) \leq F_X(y)$ for $x \leq y$;

(d) $\mathrm{P}\left(X > x\right) = 1 - F_X(x)$

(e) If $x < y$, then $\mathrm{P}\left(x < X \leq y\right) = F_X(y) - F_X(x)$

Expectation

Let X be a discrete random variable for which $\sum_{x \in D_X} |x| \, f_X(x) < \infty$, where D_X is the support of f_X.
The **expectation** (or **expected value** or **mean**) of X is

$$\mathrm{E}\left[X\right] = \sum_{x \in D_X} x \, \mathrm{P}\left(X = x\right) = \sum_{x \in D_X} x \, f_X(x).$$

Expected value of a function

Let X be a discrete random variable with mass function f, and let g be a real-valued function of $\mathbb{R}$. Then

$$\mathrm{E}\left[g(X)\right] = \sum_{x \in D_X} g(x) \cdot f(x),$$

when $\sum_{x \in D_X} |g(x)|\, f(x) < \infty$.

Properties of the expected value

Let X be a discrete random variable with expected value $\mathrm{E}[X]$, and let $a,\ b \in \mathbb{R}$ be constants. Then

(a) $\mathrm{E}[\cdot]$ is a linear operator, i.e., $\mathrm{E}[aX + b] = a\,\mathrm{E}[X] + b$;

(b) if $g(X)$ and $h(X)$ have finite expected values, then

$$\mathrm{E}[g(X) + h(X)] = \mathrm{E}[g(X)] + \mathrm{E}[h(X)] ;$$

(c) if $\mathrm{P}(X = b) = 1$, then $\mathrm{E}[X] = b$;

(d) if $\mathrm{P}(a < X \leq b) = 1$, then $a < \mathrm{E}[X] \leq b$;

(e) $\{\mathrm{E}[X]\}^2 \leq \mathrm{E}\left[X^2\right]$

Remark: Facts (a), (b) and (c) are very useful in calculations.

Example 11. Expectation

We roll two fair dice, one red and one green. Let R and G be the RV representing the value of the side facing up for the red and green dice, respectively. Let X be the RV representing the sum of the side-up faces of both dice.

(a) What is the expected value of the variables R and G?

(b) What is the expected value of X?

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Solution of Example 11

(a) $\mathrm{E}\left[R\right] = \mathrm{E}\left[G\right] = \sum_{i=1}^{6} x_i \, \mathrm{P}\left(X = x_i\right) = \frac{7}{2} = 3.5$

(b) Using the fact that expectation is linear :

$$\mathrm{E}\left[X\right] = \mathrm{E}\left[R + G\right] = \mathrm{E}\left[R\right] + \mathrm{E}\left[G\right] = 7$$

Moments of a distribution

If X has a PMF $f(x)$ such that $\sum_x |x|^r f(x) < \infty$, then

(a) the r **th moment** of X is $\mathrm{E}[X^r]$;

(b) the r **th central moment** of X is $\mathrm{E}[(X - \mathrm{E}[X])^r]$;

(c) the **variance** of X is $\sigma^2 = \mathrm{Var}[X] = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right]$ (the second central moment);

(d) the **standard deviation** of X is defined as $\sigma = \sqrt{\mathrm{Var}[X]}$ (non-negative);

Moments of a distribution: Remarks

- $\mathrm{E}[X]$ and $\mathrm{Var}[X]$ are the most important moments: they represent the **"average value"** $\mathrm{E}[X]$ of X, and the **"average squared distance"** of X from its mean, $\mathrm{E}[X]$.
- The **variance** measures the scatter of X around its mean, $\mathrm{E}[X]$, with small variance corresponding to small scatter, and conversely.
- The expectation and standard deviation have the same units (kg, m,...) as X.

Properties of the Variance

Let X be a random variable whose variance exists, and let a, b be constants. Then:

(a) $\mathrm{Var}\left[X\right] = \mathrm{E}\left[X^2\right] - \mathrm{E}\left[X\right]^2 = \mathrm{E}\left[X(X-1)\right] + \mathrm{E}\left[X\right] - \mathrm{E}\left[X\right]^2$
The variance expressed in terms of either the ordinary moments, or the factorial moments. Usually, the first is more useful, but the second can be used occasionally.

(b) $\mathrm{Var}\left[aX + b\right] = a^2\,\mathrm{Var}\left[X\right]$
The variance does not change if X is shifted by a fixed quantity b, but the dispersion is increased by the squared of a multiplier a.

(c) $\mathrm{Var}\left[X\right] = 0 \Rightarrow X$ is constant with probability 1.
If X has zero variance, then it does not vary.

Example 12. Variance

We roll a fair dice. Let X be the RV representing the value of the side-up face (the outcome). Calculate the variance of X.

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Solution of Example 12

As seen in Example 11 (a), all possible outcomes have equal probability $\frac{1}{6}$, and the expected value of the outcome corresponds to $E(X) = \frac{7}{2}$. The variance can thus be calculated as

$$Var[X] = \mathrm{E}\left[(X - E(X))^2\right] = \sum_{x=1}^{6} \frac{1}{6}\left(x - \frac{7}{2}\right)^2 = \frac{2}{6} \cdot \frac{1}{4} \cdot (1+9+25) = \frac{35}{12}$$

Questions?

Introduction to Continuous Probability Theory

Continuous random variables

In many situations, we must work with continuous variables:

- the time until the end of the lecture $\in (0, 45)$ min;
- the pair (height, weight) $\in (0, \infty)^2$.

Until now, we supposed that the support

$$D_X = \{x \in \mathbb{R} : X(\omega) = x,\, \omega \in \Omega\}$$

of X is countable, so X is a discrete random variable. We suppose now that D_X is not countable, which implies that Ω itself is not countable.

Probability density functions

A random variable X is **continuous** if there exists a function $f(x)$, called the **probability density function (or density) (PDF)** of X, such that

$$\mathrm{P}\left(X \leq x\right) = F(x) = \int_{-\infty}^{x} f(u)\,du, \quad x \in \mathbb{R}.$$

The properties of F imply that

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$

Probability density functions: Remarks

- Evidently,

$$f(x) = \frac{dF(x)}{dx}.$$

- Since $\mathrm{P}\left(x < X \leq y\right) = \int_x^y f(u)\,du$ for $x < y$, for all $x \in \mathbb{R}$,

$$\begin{aligned}\mathrm{P}\left(X = x\right) &= \lim_{y \downarrow x} \mathrm{P}\left(x < X \leq y\right) \\ &= \lim_{y \downarrow x} \int_x^y f(u)\,du \\ &= \int_x^x f(u)\,du = 0.\end{aligned}$$

- If X is discrete, then its PMF $f(x)$ is often also called its density function.

Uniform distribution

The random variable U having density

$$f(u) = \begin{cases} \frac{1}{b-a}, & a \le u \le b, \\ 0, & \text{otherwise,} \end{cases} \qquad a < b$$

is called a **uniform random variable**. We write $U \sim U(a, b)$.

Example 13. Uniform distribution

Find the cumulative distribution function (CDF) of the uniform distribution.

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Solution of Example 13

The integration of the uniform density gives

$$F(u) = \begin{cases} 0, & u \leq a, \\ \frac{u-a}{b-a}, & a < u \leq b, \\ 1, & u > b. \end{cases}$$

Moments

Let $g(x)$ be a real-valued function, and X a continuous random variable of density $f(x)$. Then if $\mathrm{E}\left[|g(X)|\right] < \infty$, we define the **expectation** of $g(X)$ to be

$$\mathrm{E}\left[g(X)\right] = \int_{-\infty}^{+\infty} g(x) f(x)\, dx.$$

In particular, the **expectation** and the **variance** of X are

$$\begin{aligned} \mathrm{E}\left[X\right] &= \int_{-\infty}^{+\infty} x f(x)\, dx, \\ \mathrm{Var}\left[X\right] &= \int_{-\infty}^{+\infty} \{x - \mathrm{E}\left[X\right]\}^2 f(x)\, dx = \mathrm{E}\left[X^2\right] - \mathrm{E}\left[X\right]^2. \end{aligned}$$

Example 14. Moments

Calculate the expectation and the variance of the uniform distribution.

Solution of Example 13

Note that we need to compute $\mathrm{E}\left[U^r\right]$ for $r = 1, 2$, and this is $\frac{1}{r+1}\frac{\left(b^{r+1}-a^{r+1}\right)}{b-a}$. Hence

$$\mathrm{E}\left[X\right] = \frac{1}{2}\frac{b^2 - a^2}{b - a} = \frac{b + a}{2}$$

as expected. For the variance, note that

$$\begin{aligned}\mathrm{E}\left[X^2\right] - \mathrm{E}\left[X\right]^2 &= \frac{1}{3}\frac{b^2 - a^3}{b - a} - \frac{(b + a)^2}{4} \\ &= \frac{1}{3}b^2 + ab + a^2 - \frac{\left(b^2 + 2ab + a^2\right)}{4} \\ &= \frac{(b - a)^2}{12}\end{aligned}$$

X DISCRETE OR CONTINUOUS?

	Discrete	Continuous
Support D_X	countable	contains an interval $(x_-, x_+) \subset \mathbb{R}$
f_X	mass function dimensionless $0 \leq f_X(x) \leq 1$ $\sum_{x \in \mathbb{R}} f_X(x) = 1$	density function units $[x]^{-1}$ $0 \leq f_X(x)$ $\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$
$F_X(a) = \mathrm{P}\,(X \leq a)$	$\sum_{x \leq a} f_X(x)$	$\int_{-\infty}^{+\infty} f_X(x)\, dx$
$\mathrm{P}\,(X \in \mathcal{A})$	$\sum_{x \in \mathcal{A}} f_X(x)$	$\int_{\mathcal{A}} f_X(x)\, dx$
$\mathrm{P}\,(a < X \leq b)$	$\sum_{\{x: a < X \leq b\}} f_X(x)$	$\int_a^b f_X(x)\, dx$
$\mathrm{P}\,(X = a)$	$f_X(a) \geq 0$	$\int_a^a f_X(x)\, dx = 0$
$\mathrm{E}\,[g(X)]$	$\sum_{x \in \mathbb{R}} g(x) f_X(x)$	$\int_{-\infty}^{+\infty} g(x) f_X(x)\, dx$

Normal distribution (I)

A random variable X having density

$$f(x) = \frac{1}{(2\pi)^{\frac{1}{2}}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \quad x \in \mathbb{R}, \quad \mu \in \mathbb{R}, \quad \sigma > 0,$$

is a **normal random variable** with expectation μ and variance σ^2: we write $X \sim \mathcal{N}(\mu, \sigma^2)$.
Note: The standard deviation of X is $\sqrt{\sigma^2} = \sigma > 0$.

Normal distribution (II)

When $\mu = 0$, $\sigma^2 = 1$, the corresponding random variable Z is **standard normal**, $Z \sim \mathcal{N}(0,1)$, with density

$$\phi(z) = (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{z^2}{2}\right), \quad z \in \mathbb{R}.$$

Then

$$F_Z(x) = \mathrm{P}\left(Z \leq x\right) = \Phi\left(x\right) = \int_{-\infty}^{x} \phi(z)\, dz = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{x} \exp \frac{-z^2}{2}\, dz$$

Note that $f(x) = \sigma^{-1}\phi(\frac{x-\mu}{\sigma})$ for $x \in \mathbb{R}$.

Questions?

Monte Carlo Integration

Integration using Deterministic Quadrature

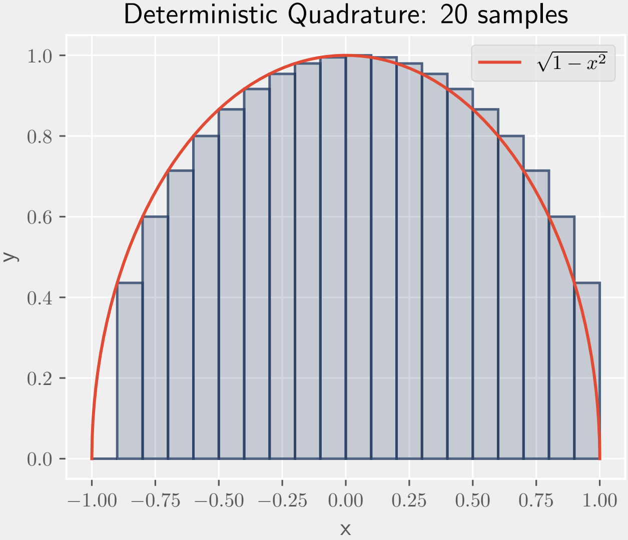
Assume we have some function $f(x)$ defined over the domain $x \in [a, b]$. We want to to evaluate the integral

$$I = \int_a^b f(x)\,dx.$$

We can approximate this integral using a deterministic quadrature rule which computes the sum of the area of regions (possibly uniformly spaced) over the domain as follow:

$$I \approx \sum_{i=1}^{N} w_i f(x_i) = \sum_{i=1}^{N} \frac{f(x_i)(b-a)}{N}$$

Integration using Deterministic Quadrature



Integration using Monte Carlo

The Monte Carlo approach to computing the integral is to consider N samples to estimate the value of the integral. The samples are selected randomly over the domain of the integral with probability density function $p(x)$.

In it's simplest form $p(x)$ can simply follow a uniform random distribution, that is $X \sim U(a, b)$, where $[a, b]$ is the domain where the function is defined.

Estimator

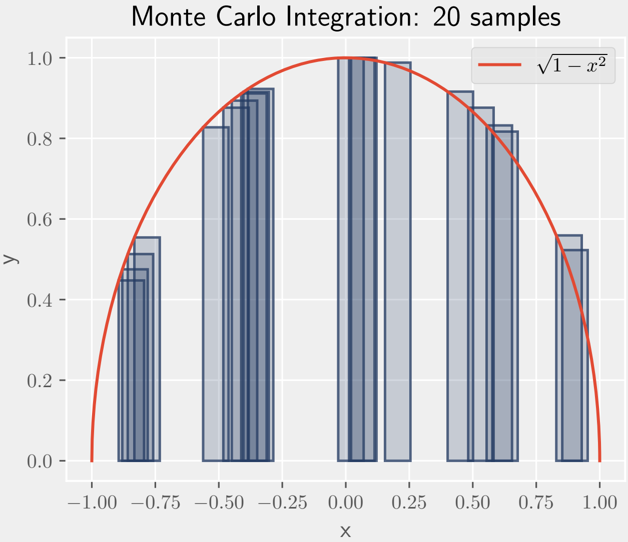
Given a random variable X with probability density function $p_X(x) = p(x)$, a function $f(x)$ to integrate and N samples x_i (or realisations) of the random variable X, we can compute the **estimator** $\langle I \rangle$ of the integral $I = \int_a^b f(x)\,\mathrm{d}x$ as:

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(x_i)}{p(x_i)}.$$

Estimator : Proof

$$
\begin{aligned}
\mathrm{E}\left[\langle I\rangle\right] &= \mathrm{E}\left[\frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p\left(x_i\right)}\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\mathrm{E}\left[\frac{f(x_i)}{p\left(x_i\right)}\right] \\
&= \frac{1}{N}N\int\frac{f(x)}{p(x)}p(x)\mathrm{d}x \\
&= \int f(x)\mathrm{d}x \\
&= I
\end{aligned}
$$

Integration using Monte Carlo : $X \sim U(a, b)$



Importance Sampling

The biggest advantage of Monte Carlo integration compared to quadrature approaches is that it only needs a **fixed number of samples regardless of the dimensionality** of the function we integrate. For example for a 2-dimensional function $f(x, y)$ we can simply its integral

$$I = \iint f(x, y)\mathrm{d}x\mathrm{d}y$$

using the estimator

$$\langle I \rangle = \frac{1}{N}\sum_{i=1}^{N} \frac{f(x_i, y_i)}{p\left(x_i, y_i\right)}$$

Integration using Monte Carlo

For now, we used a simple uniform distribution, which can lead to **high variance** in the estimator. Ideally, we want the density function $p(x) \propto f(x)$. Then, a single sample would suffice to estimate the constant proportionality factor and $\mathrm{Var}\left[\langle I \rangle\right] = 0$. This is called **perfect importance sampling**.

Of course, this is often not feasible since finding the ideal $p(x)$ might be as hard as computing the integral of $f(x)$. However, if $p(x)$ is a good approximation of $f(x)$ the variance of our estimator would already greatly decrease.

We call this **importance sampling**, since $p(x)$ should put more weight (or importance) where the function $f(x)$ takes large values and less weight to lower values of $f(x)$.

Sampling from a given distribution $p(x)$

Given a probability density function $p(x)$ and a uniformly sampled number $U \sim \mathrm{U}(0, 1)$ we can sample $X \sim p(x)$ using the following pseudo-code:

- Pick u unifomly in $[0, 1)$
- Output $x = F^{-1}(u)$

Example 15. Inverse CDF Computation I

Suppose we want to take samples proportional to $g(x) = \cos(\frac{\pi}{2}x)$ and $x \in [-1, 1]$. First, we need to normalize $g(x)$ to turn it into a valid probability density function:

$$p_X(x) = \frac{g(x)}{\int_{-1}^{1} g(x)\mathrm{d}x} = \frac{\pi}{4}\cos\left(\frac{\pi}{2}x\right) \tag{1}$$

Then we can compute its CDF as follow:

$$F_X = \int_{-1}^{x} p_X(x)\mathrm{d}x = \frac{1}{2}\left(\sin\left(\frac{\pi}{2}x\right) + 1\right) \tag{2}$$

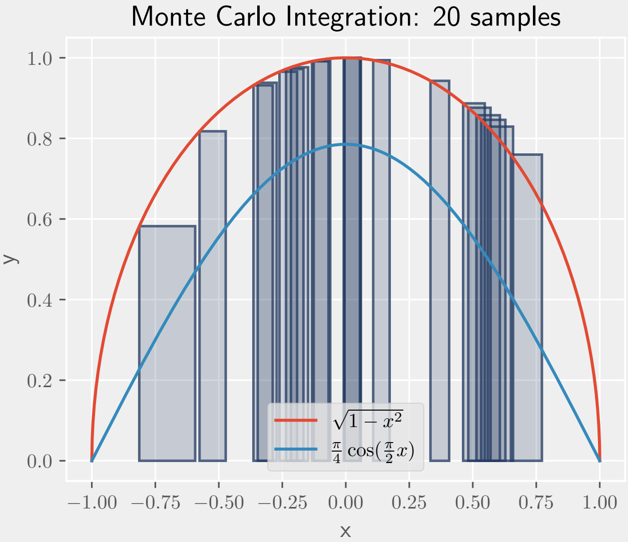
Example 15. Inverse CDF Computation II

And the inverse CDF is:

$$F_X^{-1}(x) = \frac{2}{\pi} \sin^{-1}(2x - 1)$$

Hence, given a uniform number $U \sim \mathrm{U}(0, 1)$ we can generate $X \sim p(x)$ using $X = F_X^{-1}(U)$

Integration using Monte Carlo : $X \sim p(x)$



The importance of Random Number generation

Generating a uniform random number in $[0, 1)$ on a computer is a long standing problem in computer science.
A good quality random number generator should exibit the following properties:

- Uniform Distribution
- Independence
- Reproducibility
- Statistical Properties
- Long Period
- Fast Generation
- Security

Monte Carlo for Rendering

In rendering the function we are interested in integrating is called the **Rendering Equation** (more in the next lecture):

$$L\left(x, \omega_o\right) = L_e\left(x, \omega_o\right) + \int_{\Omega_+} f_r\left(\omega_i, x, \omega_o\right) L_i\left(x, \omega_i\right) \cos(\theta_i)\, \mathrm{d}\omega_i \qquad (3)$$

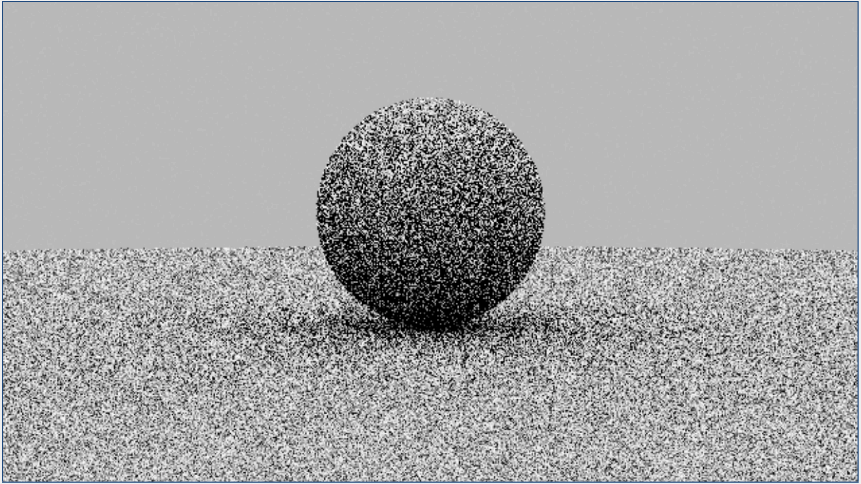
Monte Carlo integration is well suited for this very high dimensional integral.

Finding a probability density that matches the entire rendering equation at once is extremely hard, however, we can still find good probability density functions for the individual steps.

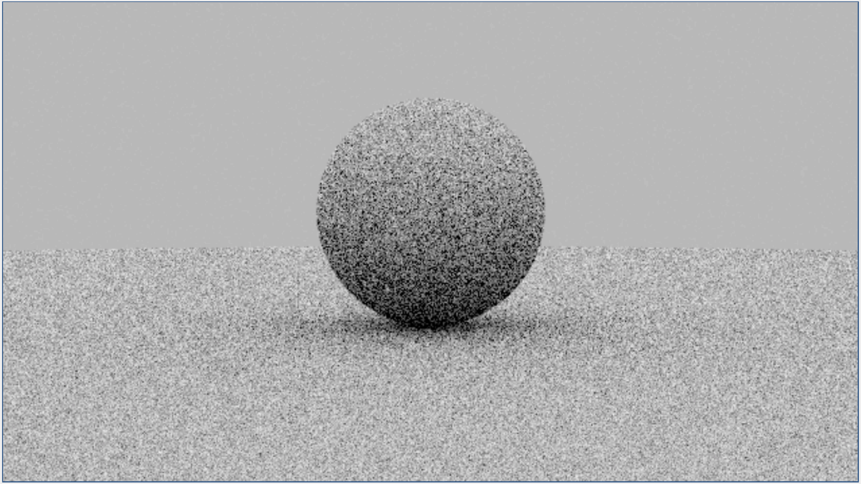
Often, for sampling directions, we want to consider the material reflection term $f_r\left(\omega_i, x, \omega_o\right)$ together with the cosine term $\cos(\theta_i)$ separately from the incident light term $L_i\left(x, \omega_i\right)$.

While it is not always easy to find a directional density function that is proportional to the reflectance function we know the direction should at least be proportional to the cosine term!

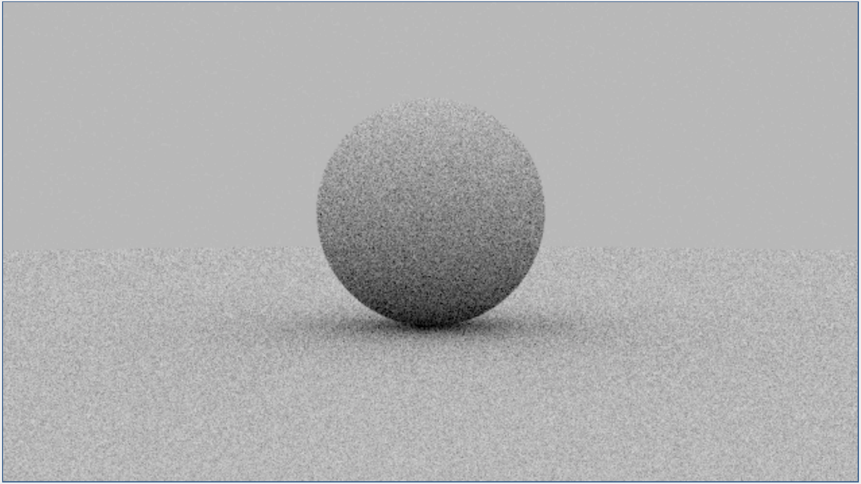
Uniform Hemisphere Sampling : 1 Sample



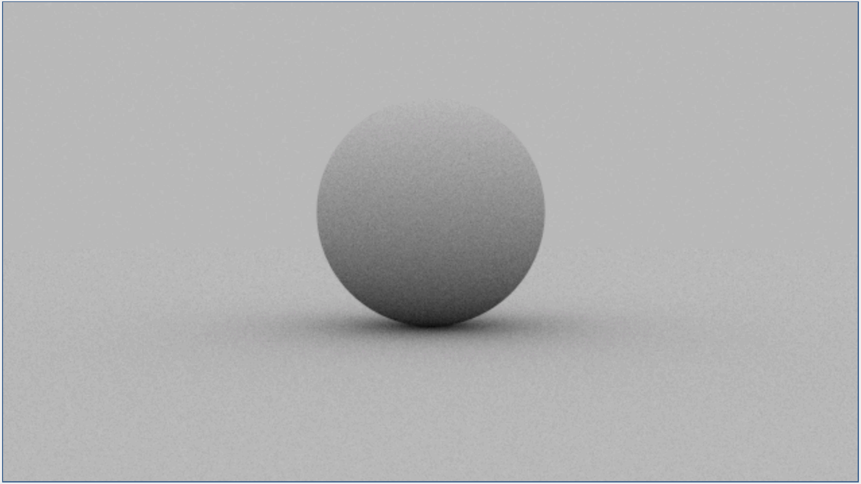
Uniform Hemisphere Sampling : 4 Samples



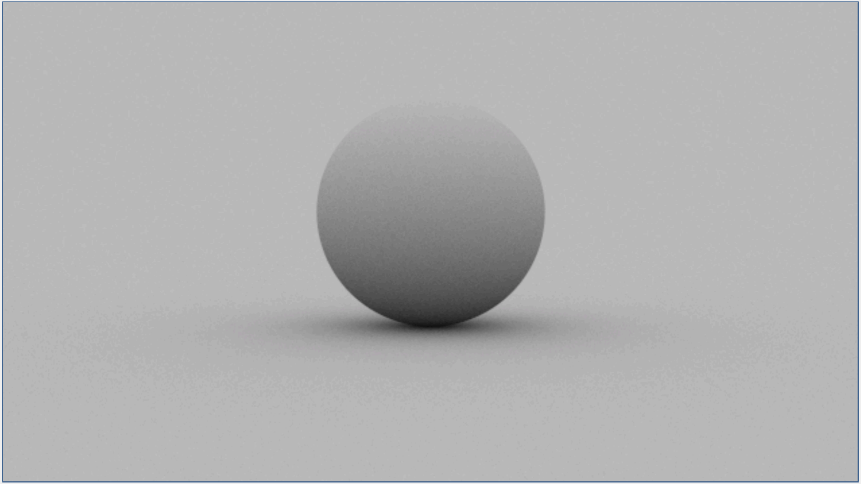
Uniform Hemisphere Sampling : 16 Samples



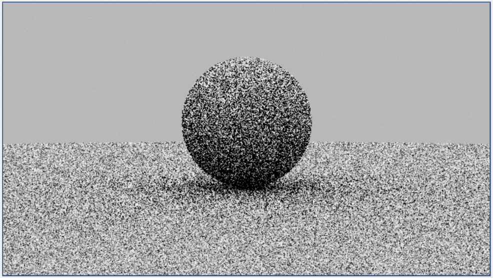
Uniform Hemisphere Sampling : 256 Samples

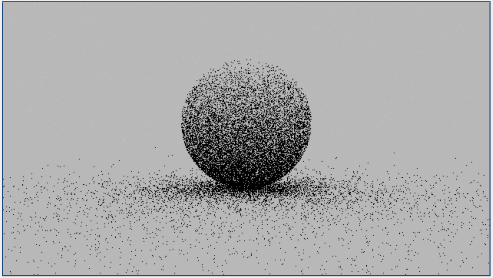


Uniform Hemisphere Sampling : 1024 Samples

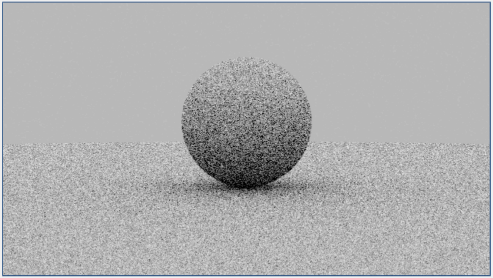


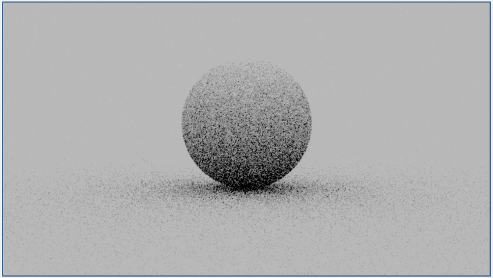
VS COSINE HEMISPHERE SAMPLING : 1 SAMPLE



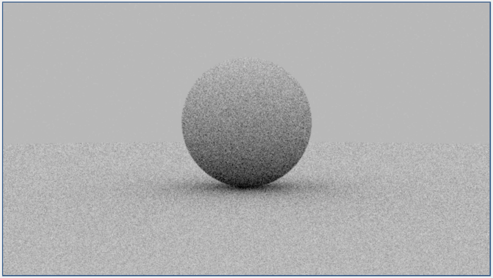


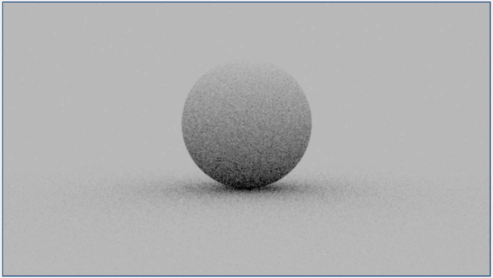
VS Cosine Hemisphere Sampling : 4 Samples



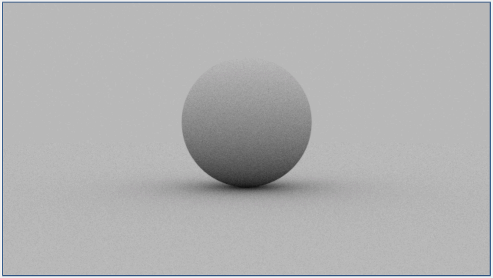


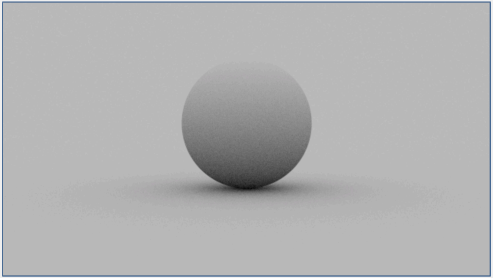
VS Cosine Hemisphere Sampling : 16 Samples



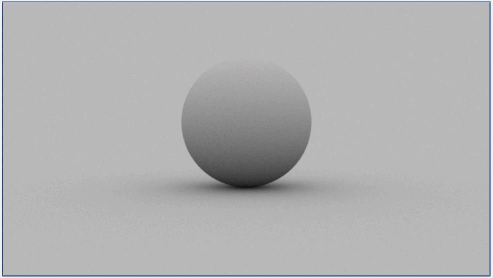


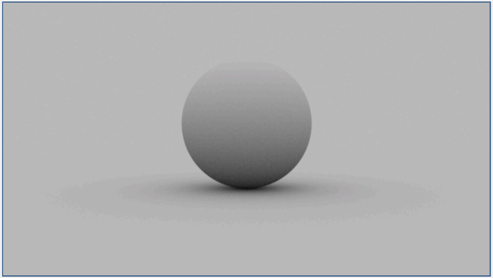
VS Cosine Hemisphere Sampling : 256 Samples





VS Cosine Hemisphere Sampling : 1024 Samples





Questions?

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