Computer Graphics
Spectral Analysis

Miša Korać
Philippe Weier
Alexander Rath
Philipp Slusallek

COMPUTER GRAPHICS
SAARLAND UNIVERSITY

November 30th, 2023

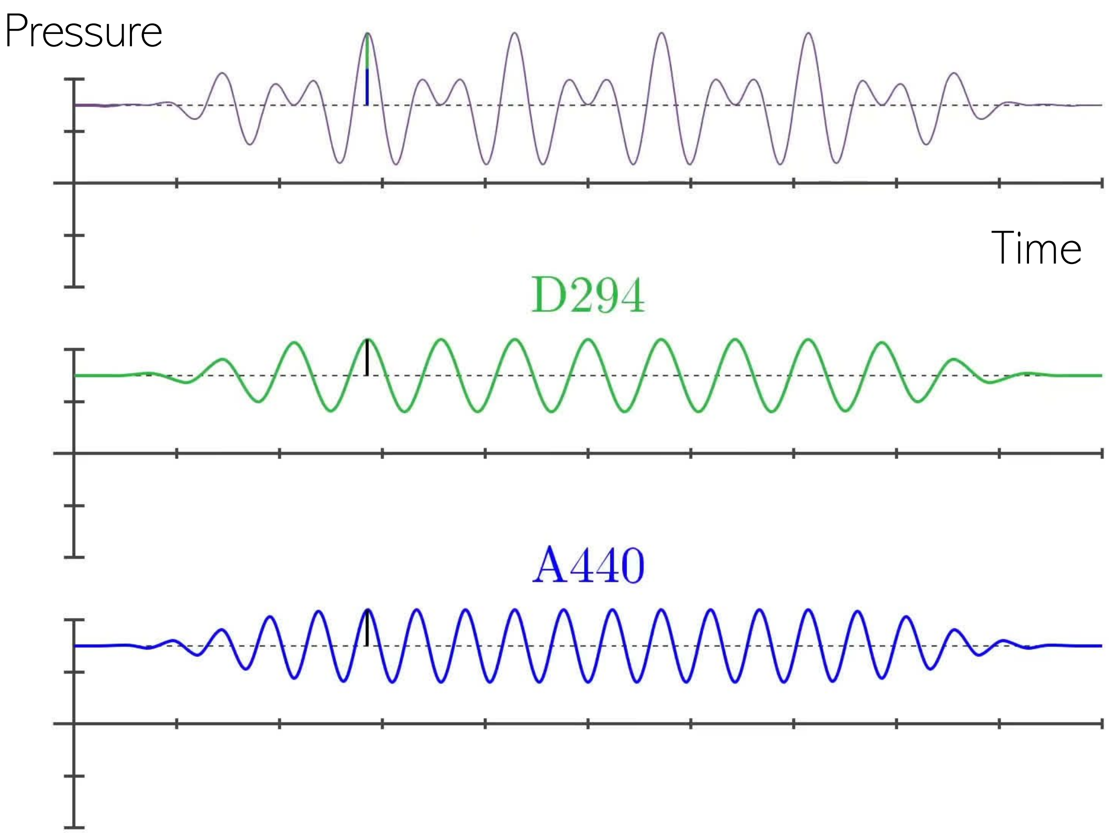
Motivation

Image Processing and Rendering

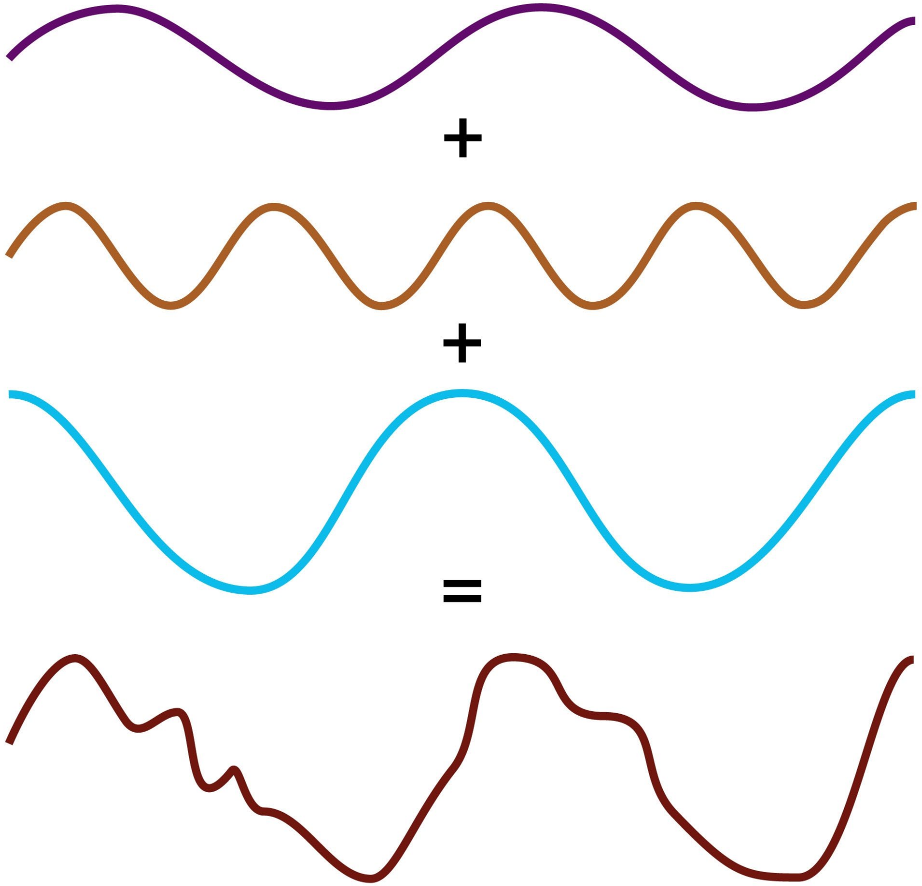
[Egan et al. 2009]

Digital Signal Procesing

Fourier Transformation: Audio Signal Analogy



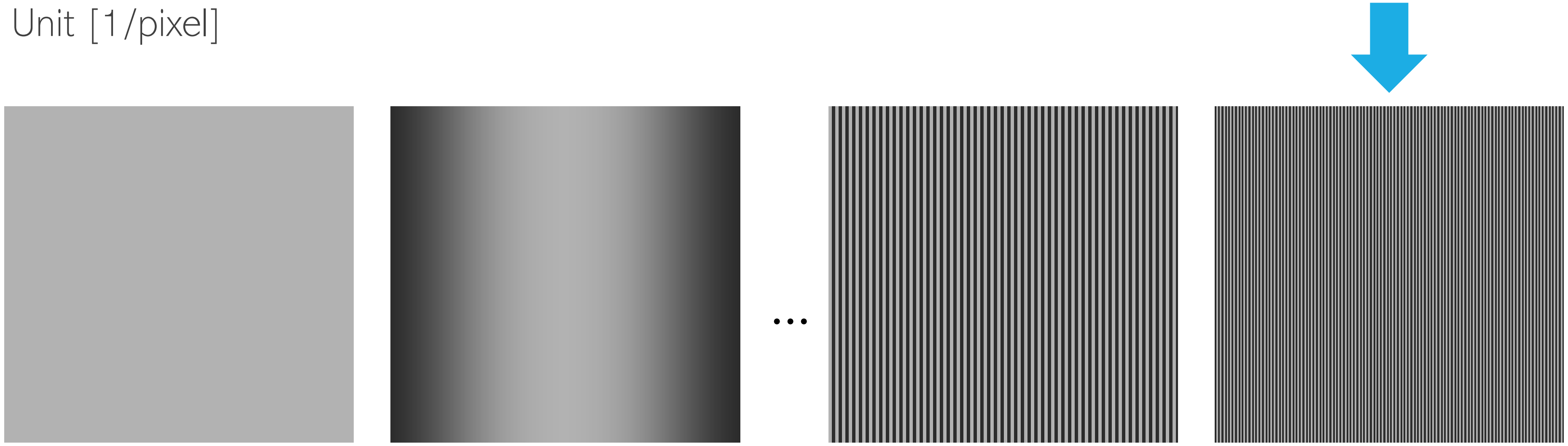
Fourier Transformation



Spatial Frequency

- Inverse of period length of some structure in an image
- Unit [1/pixel]

Nyquist frequency



Lowest frequency
Image average

Highest frequency
½ of image resolution

Spatial Frequency

Fourier Transformation

Analysis:
Fourier Transformation

$$F(k) = F_x[f(x)](k) = \int_{-\infty}^{\infty} f(x)e^{-i2\pi kx}dx$$

Synthesis:
Inverse Fourier Transformation

$$f(x) = F_x^{-1}[F(k)](x) = \int_{-\infty}^{\infty} F(k)e^{i2\pi kx}dk$$

Representation via complex exponential:

- ***e^{ix} = cos(x) + i sin(x)*** (see Taylor expansion)
- Use to describe phase information: shifting of the pattern.

Division into odd and even parts

Division into even and odd parts

- Even: f(x) = f(-x) (symmetry about *y* axis): Described by cosine
- Odd: f(x) = -f(-x) (symmetry about origin): Described by sine.

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] = E(x) + O(x)$$

Analysis

$$F(k) = \int_{-\infty}^{\infty} f(x)\,(\cos(-2\pi kx) + i\sin(-2\pi kx))dx = b(k) - i\,a(k)$$

Even term

$$b(k) = \int_{-\infty}^{\infty} f(x)\cos(2\pi kx)\,dx = \int_{-\infty}^{\infty} (E(x) + O(x))\cos(2\pi kx)\,dx = \int_{-\infty}^{\infty} E(x)\cos(2\pi kx)\,dx$$

Odd term

$$a(k) = \int_{-\infty}^{\infty} f(x)\sin(2\pi kx)\,dx = \int_{-\infty}^{\infty} (E(x) + O(x))\sin(2\pi kx)\,dx = \int_{-\infty}^{\infty} O(x)\sin(2\pi kx)\,dx$$

Synthesis

$$f(x) = \int_{-\infty}^{\infty} F(k)(\cos(2\pi kx) + i\sin(2\pi kx))\,dk = E(x) + O(x)$$

Even term

$$E(x) = \int_{-\infty}^{\infty} F(k)\cos(2\pi kx)\,dk = \int_{-\infty}^{\infty} (b(k) - i\,a(k))\cos(2\pi kx)\,dk = \int_{-\infty}^{\infty} b(k)\cos(2\pi kx)\,dk$$

Odd term

$$O(x) = \int_{-\infty}^{\infty} F(k)\,i\sin(2\pi kx)\,dk = \int_{-\infty}^{\infty} (b(k) - i\,a(k))\,i\sin(2\pi kx)\,dk = \int_{-\infty}^{\infty} a(k)\sin(2\pi kx)\,dk$$

Spatial vs Frequency Domain: Important basis functions

Box ↔ (normalized) *sinc*

$$\mathrm{sinc}(x) = \frac{\sin(x\pi)}{x\pi}$$

$$\mathrm{sinc}(0) = 1$$

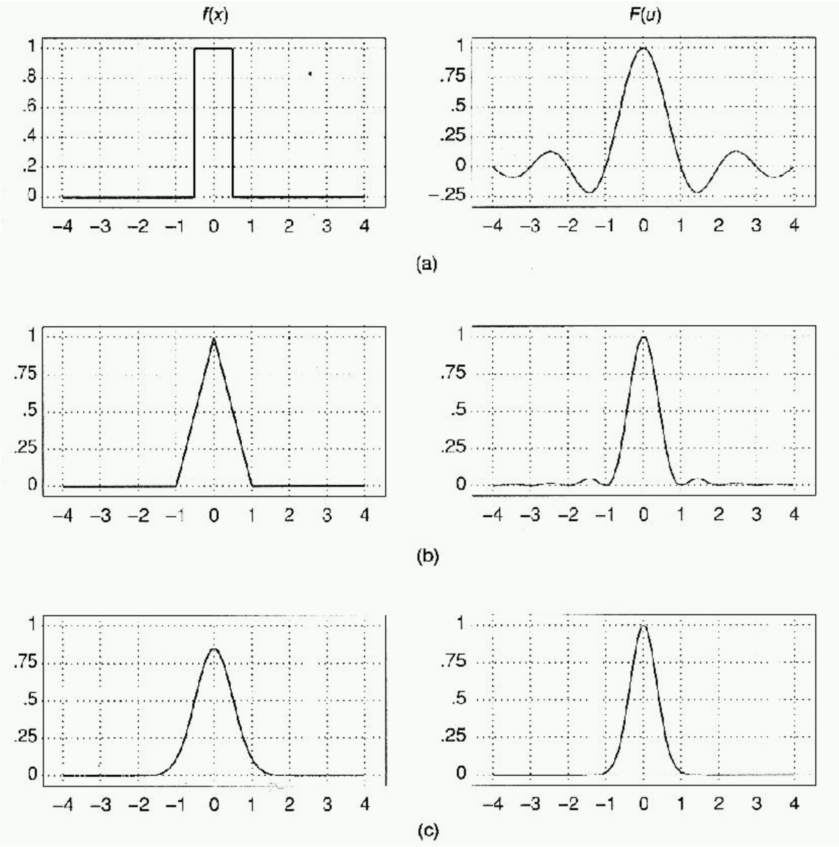
$$\int \mathrm{sinc}(x)dx = 1$$

- **Negative values**
- **Infinite support!**

Gaussian ↔ Gaussian

- Inverse width

Spatial Domain | Frequency Domain



Tent ↔ *sinc*^{2}

- *Tent is convolution of box function with itself*

Spatial vs Frequency Domain: Transform behavior

Fourier transform of a box function

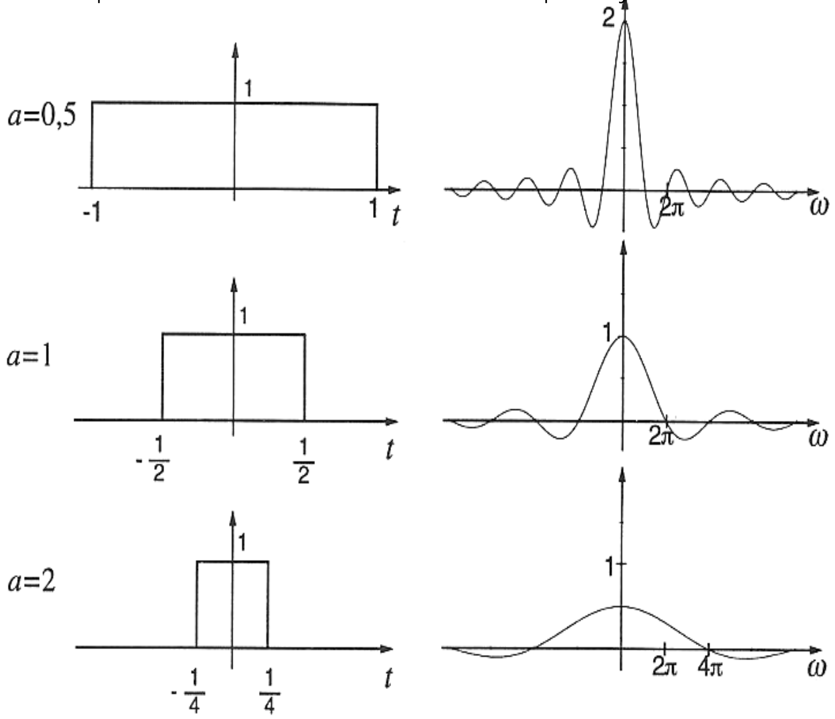
$$\mathrm{rect}(at) \;\circ\!\!-\!\!\bullet\; \frac{1}{|a|}\,\mathrm{si}\left(\frac{\omega}{2a}\right).$$

Spatial Domain | Frequency Domain

Wide box — **Narrow** *sinc*

Box — *Sinc*

Narrow box — **Wide** *sinc*

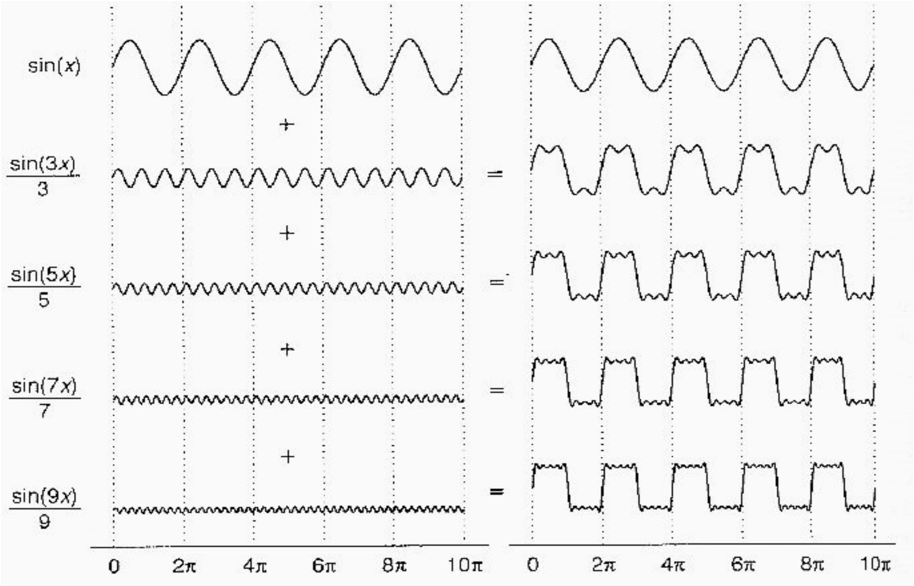


Example: Fourier Synthesis (Inverse Fourier Transformation)

- **Square wave: periodic, uneven function**

$$f(x) = 0.5 \quad \forall\, 0 < (x \bmod 2\pi) < \pi$$
$$= -0.5 \quad \forall\, \pi < (x \bmod 2\pi) < 2\pi$$
$$a_k = \int \sin(2\pi kx) f(x)\, dx \qquad f(x) = \sum_k a_k \sin(2\pi kx)$$

- $a_0 = 0$
- $a_1 = 1$
- $a_2 = 0$
- $a_3 = 1/3$
- $a_4 = 0$
- $a_5 = 1/5$
- $a_6 = 0$
- $a_7 = 1/7$
- $a_8 = 0$
- $a_9 = 1/9$
- …



Discreate Fourier Transformation

Periodic in space ⇔ discrete in frequency (vice ver.)

- Any periodic, continuous function can be expressed as the sum of an (infinite) number of sine or cosine waves:

$$f(x) = \Sigma_k a_k \sin(2\pi * k * x) + b_k \cos(2\pi * k * x)$$

Decomposition of signal into different frequency bands: spectral analysis

- Frequency band: k **(must be an integer)**
  - $k = 0$: mean value
  - $k = 1$: function period, lowest possible frequency
  - k_{max}? : band limit, no higher frequency present in signal
- Fourier coefficients: a_k, b_k (real-valued, as before)
  - Even function $f(x) = f(-x)$: $a_k = 0$
  - Odd function $f(x) = -f(-x)$: $b_k = 0$

Discrete Fourier Transformation

Equally-spaced function samples (N samples)
- Function values known only at discrete points, e.g.
  - Idealized Physical measurements
  - Pixel positions in an image!
    - Represented via sum of Delta distribution (Fourier integrals → sums)

Fourier analysis

$$a_k = \sum_i \sin\left(\frac{2\pi k i}{N}\right) f_i$$

$$b_k = \sum_i \cos\left(\frac{2\pi k i}{N}\right) f_i$$

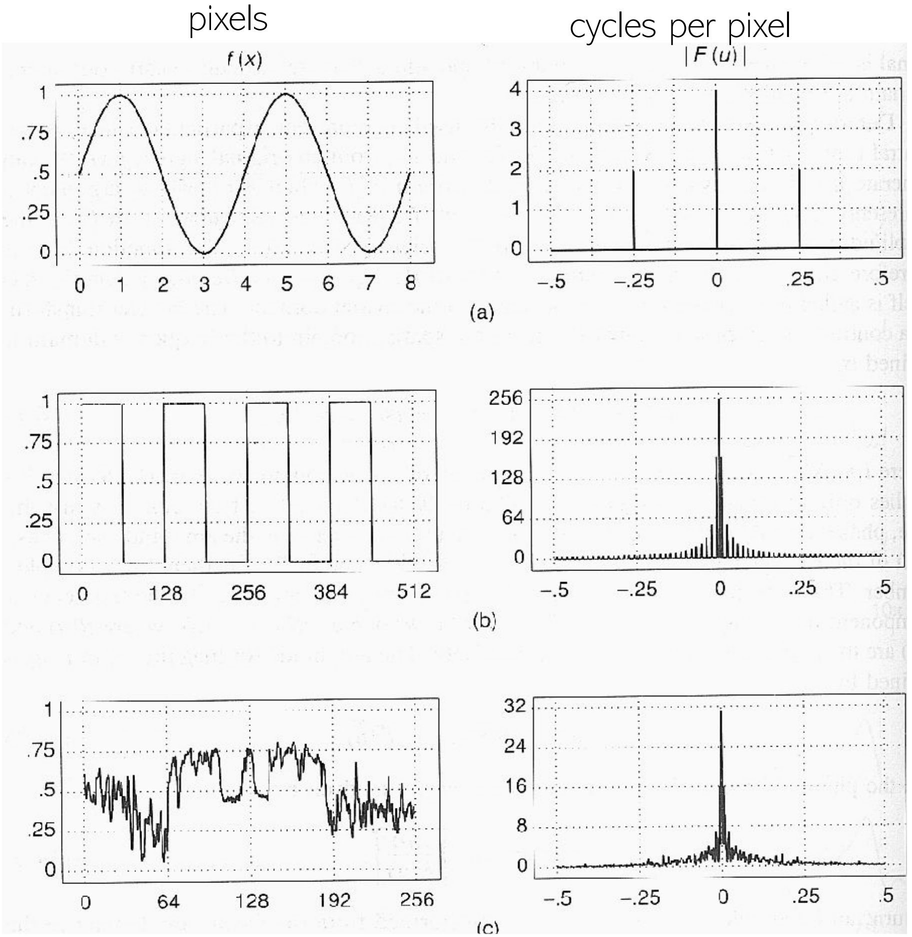
- Sum over all N measurement points
- $k = 0, 1, 2, \ldots$? Highest possible frequency?
  - **Nyquist frequency: highest frequency that can be represented**
  - **Defined as 1/2 the sampling frequency**
  - Sampling rate N: determined by image resolution (pixel size)
  - 2 samples / period ↔ 0.5 cycles per pixel ⇒ $k_{max} \leq N/2$

Spatial vs. Frequency Domain: Examples

Sine wave with positive offset

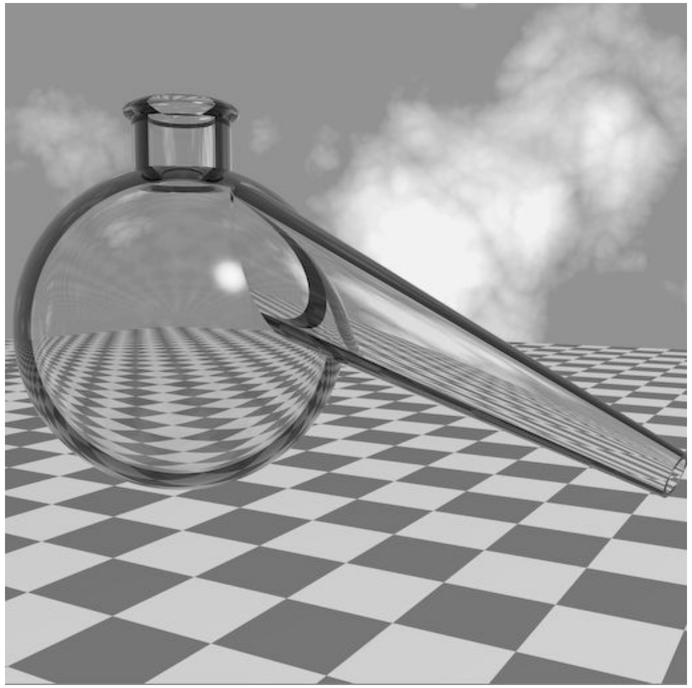
Square wave

Scanline of an image

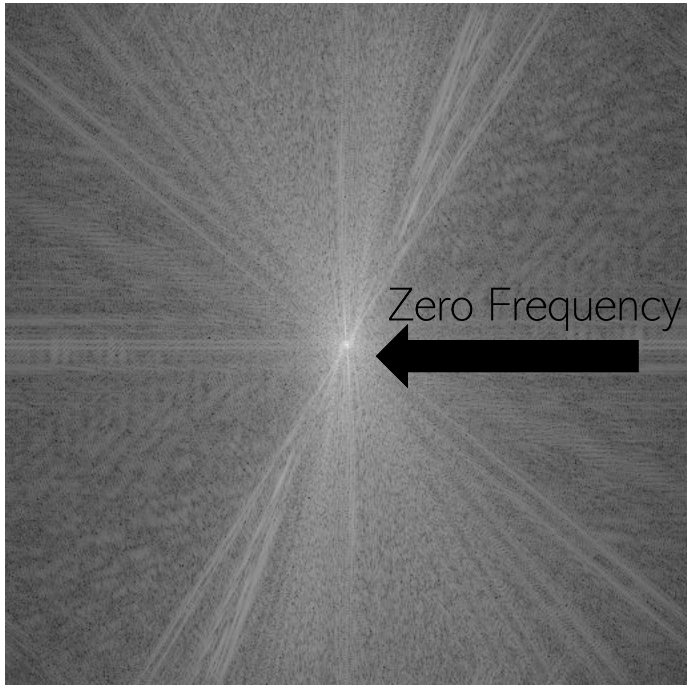


2D Fourier Transformation

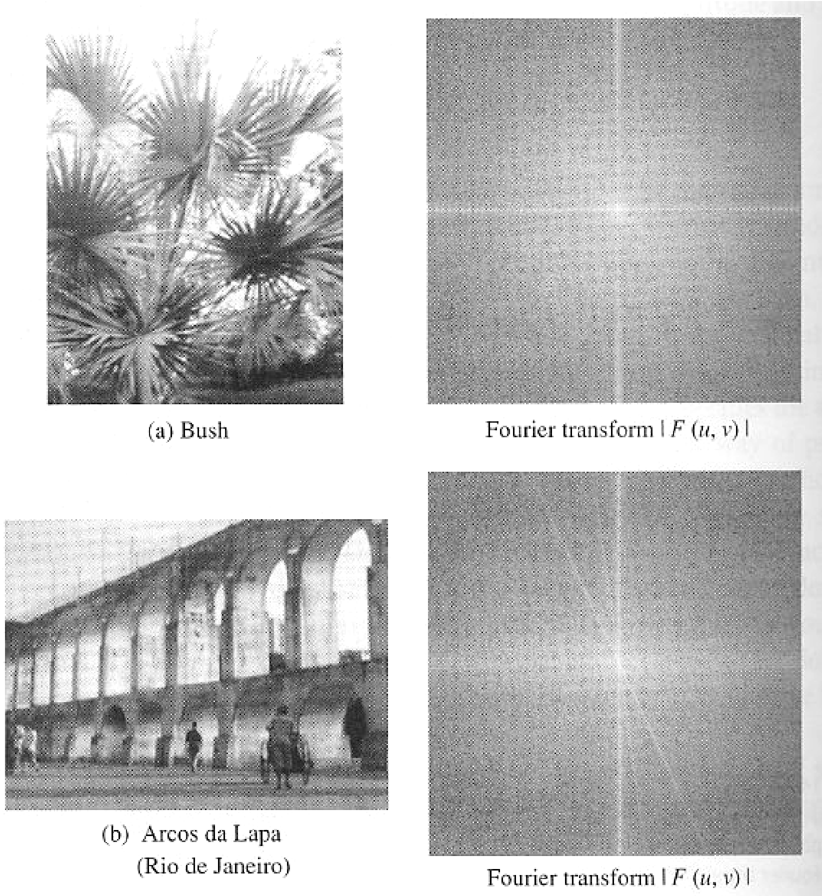
- 2D Fourier Transformation can be **separated into two 1D Fourier transformations** along x and y directions.
- **Discontinuities**: orthogonal direction in Fourier domain!



Rendered Image



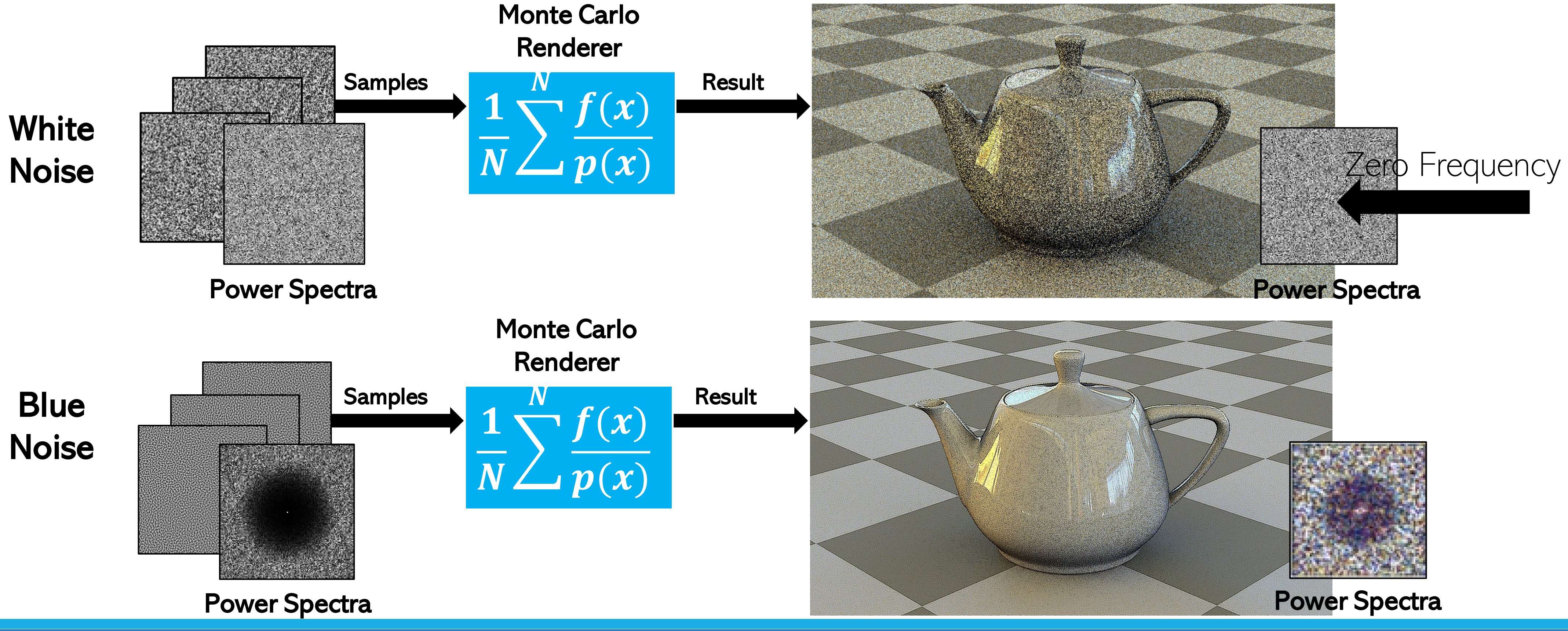
Fourier Transform



Power Spectra

With Sneak Peak into Realistic Image Synthesis

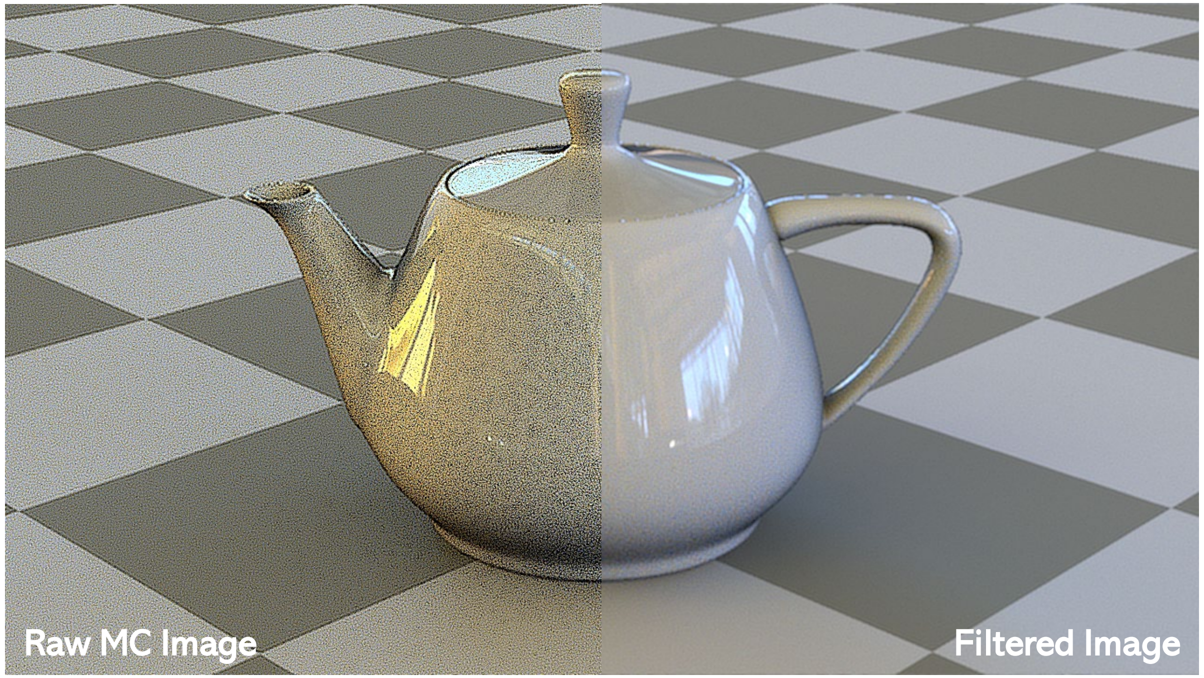
Power spectrum describes the distribution of power into frequency components.



Convolution: Motivation

Describes many natural processes:

- Room Impulse Measurement in Acoustics
- Image Processing: Filtering

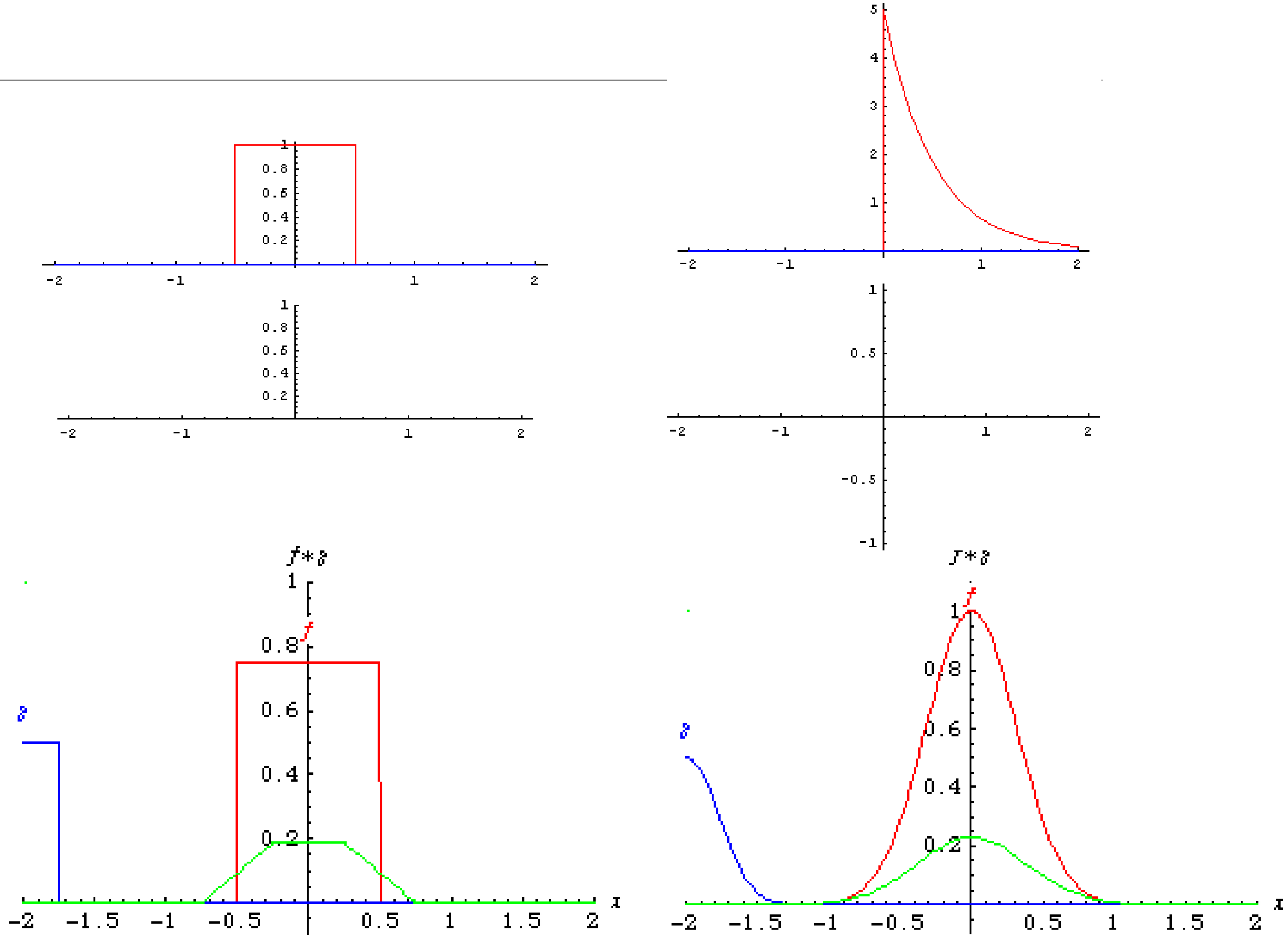


Convolution

$$(f \otimes g)(x) = \int_{-\infty}^{\infty} f(\tau)g(x-\tau)d\tau$$

Expensive operation in image space

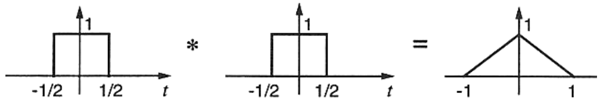
- For each x integrate over non-zero domain



Convolution: Fourier vs Image Space

Image Domain		Fourier domain
Convolution	$\rightarrow$	**Multiplication**
Multiplication	$\rightarrow$	**Convolution**

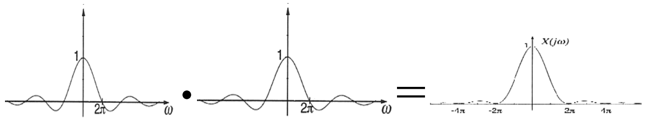
Multiplication in transformed Fourier domain is cheaper than direct convolution in image domain!



$$x(t) = \begin{cases} t+1 & \text{für } -1 \le t < 0 \\ -t+1 & \text{für } 0 \le t < 1 \\ 0 & \text{sonst} \end{cases}$$

$$\mathrm{rect}(t) * \mathrm{rect}(t) = x(t)$$

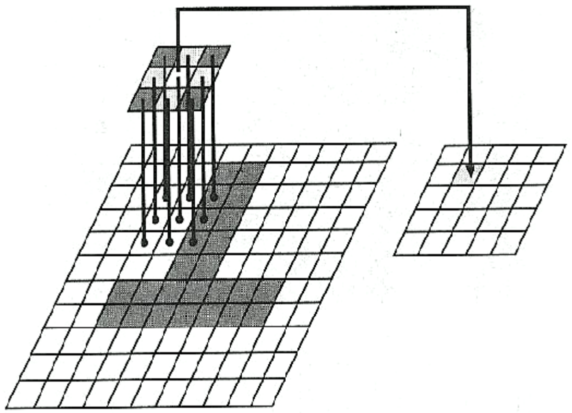
$$\mathrm{si}\left(\frac{\omega}{2}\right) \cdot \mathrm{si}\left(\frac{\omega}{2}\right) = X(j\omega) = \mathrm{si}^2\left(\frac{\omega}{2}\right).$$

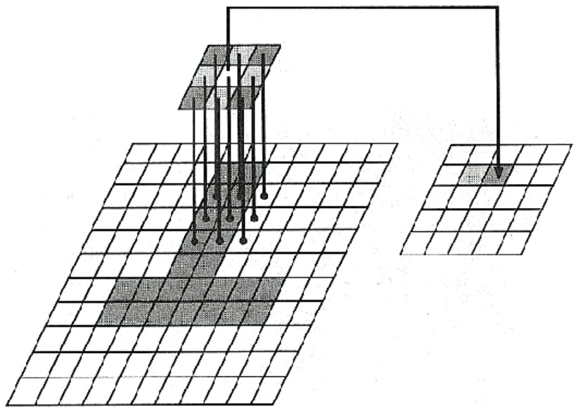


Convolution and Filtering

Technical realization
- In image domain
- Pixel mask with weights

Problems (e.g. *sinc*)
- Large filter support
  - Large mask (resolution of the image)
  - **A lot of computation**
- **Negative weights might introduce problems if not handled properly**





Filtering

Low-pass filtering

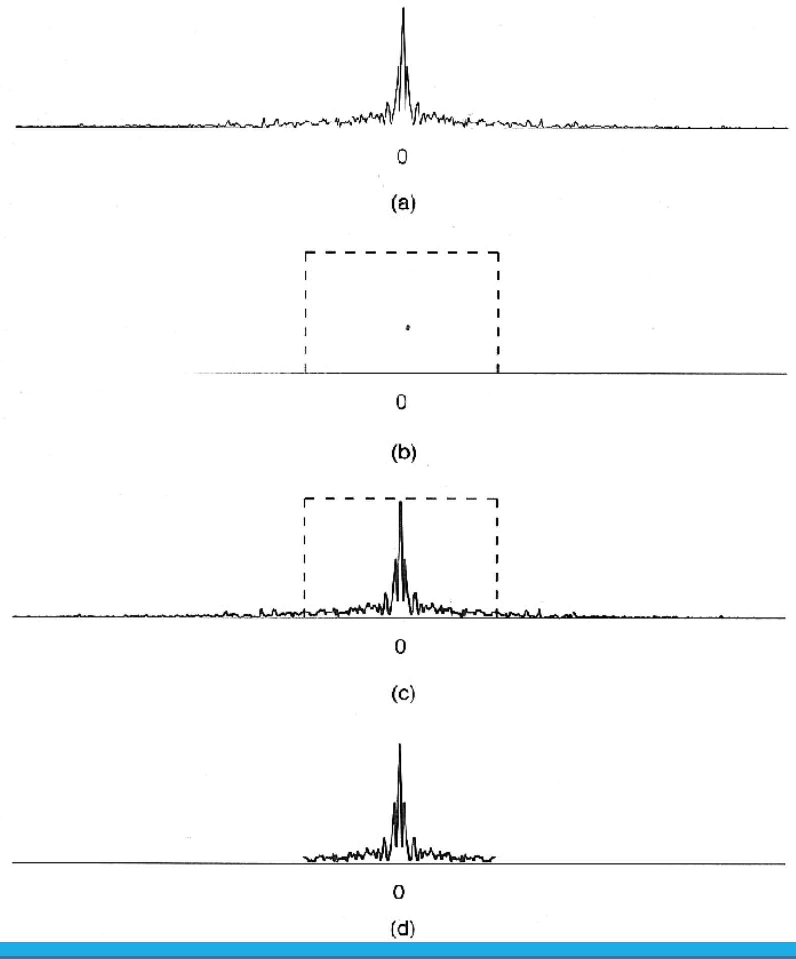
- Multiplication with box in frequency domain
- Convolution with *sinc* in spatial domain

High-pass filtering

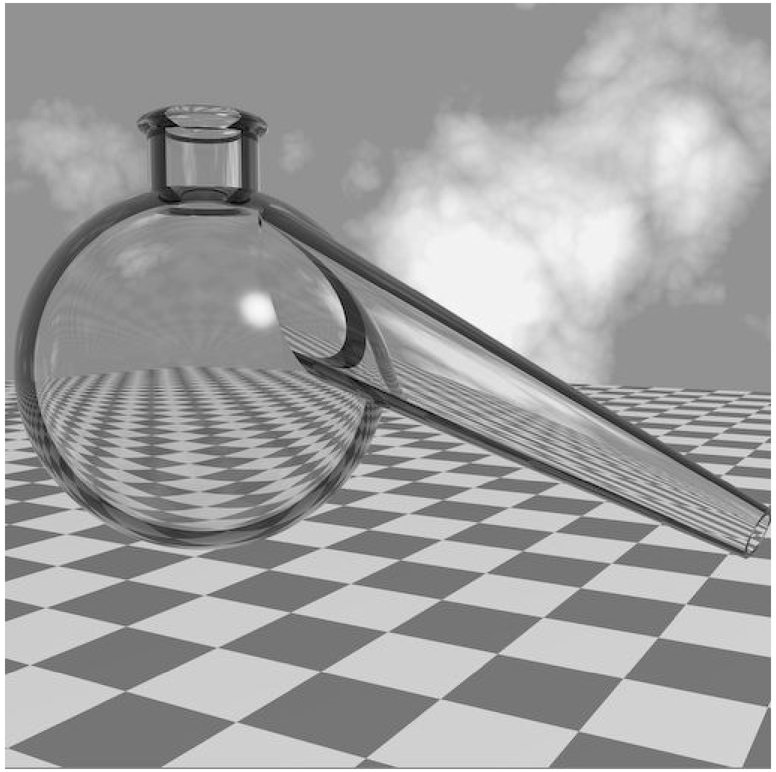
- Multiplication with (1 - box) in frequency domain
- Only high frequencies

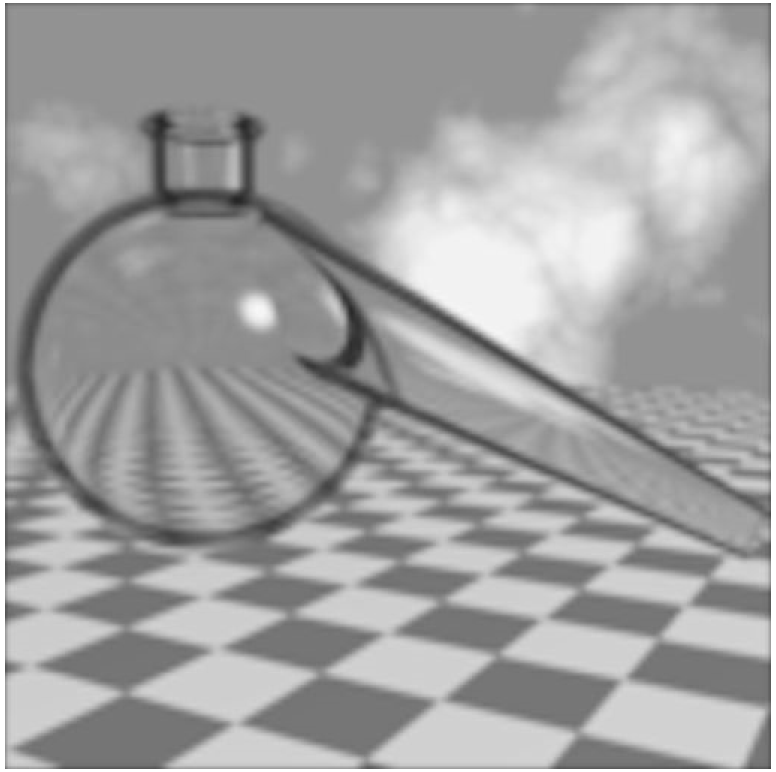
Band-pass filtering

- Only intermediate



Low Pass Filtering: Blurring

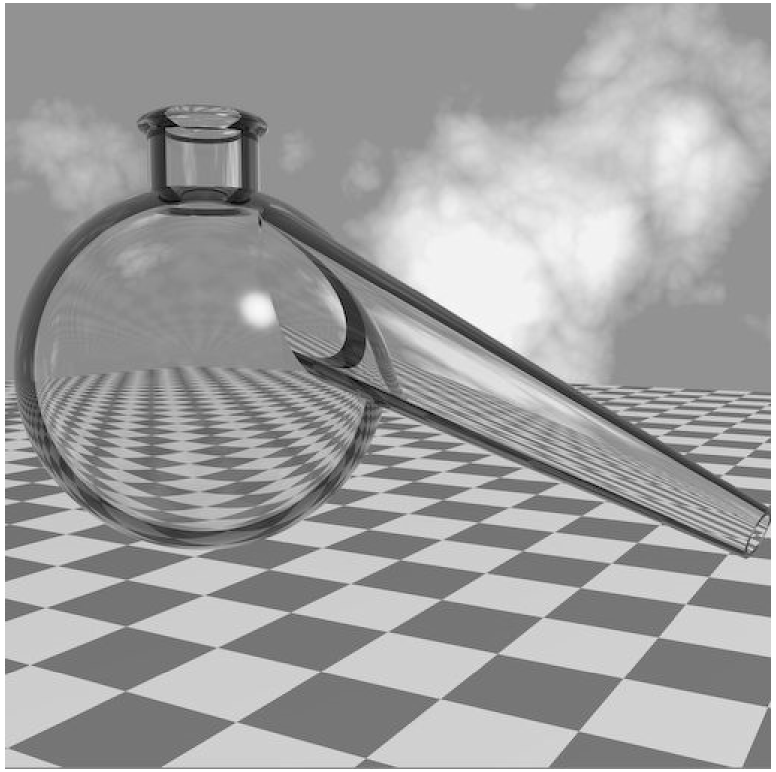


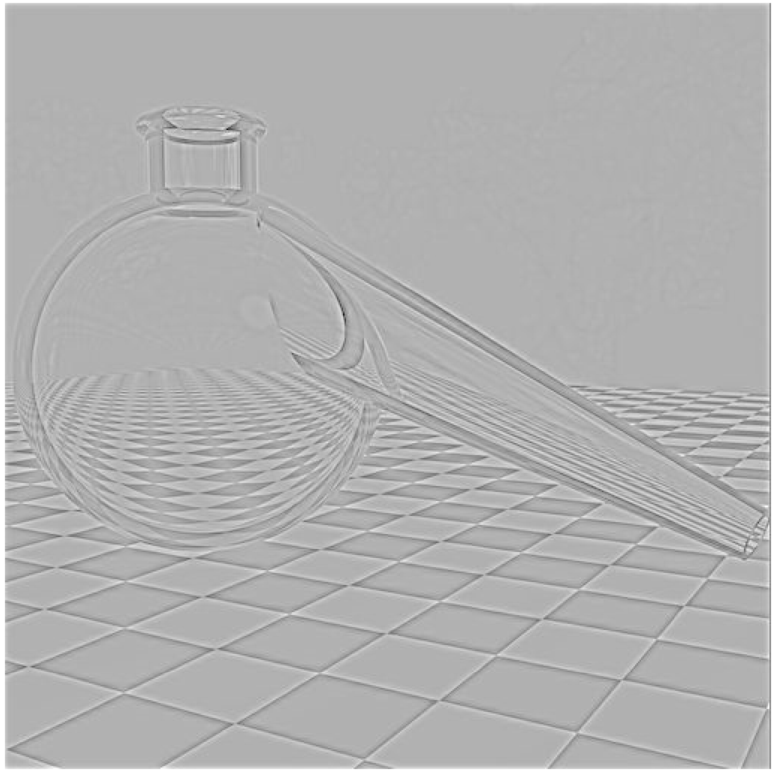


High-Pass Filtering

Enhances discontinuities in image

- Useful for edge detection





Anything Clear?